Delta-matroids and delta-matroid polynomials

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Ribbon graphs

A topological graph with
- discs for vertices,
- ribbons for edges.

- Considered up to homeomorphisms that preserve vertex-edge structure (including cyclic order of edges at vertices).

- Topologically a punctured surface.
Definition

A ⊆ E is a quasi-tree in a connected ribbon graph (V, E) if the boundary of (V, A) has one component.
Definition

$A \subseteq E$ is a quasi-tree in a connected ribbon graph $(V, E)$ if the boundary of $(V, A)$ has one component.

Example

A spanning tree is a quasi-tree.
Ribbon graphs and quasi-trees

**Definition**

A \( A \subseteq E \) is a *quasi-tree* in a connected ribbon graph \((V, E)\) if the boundary of \((V, A)\) has one component.

**Example**

A spanning tree is a quasi-tree.
What is it?

\[ R \rightarrow ? \]

\[ G \rightarrow M \]
Delta-matroid

**Definition**

$(E, B)$ form a matroid if

1. $B \neq \emptyset$.

2. If $B_1, B_2 \in B$ and $e \in B_1 \triangle B_2$ then there exists $f \in B_1 \triangle B_2$ such that $B_1 \triangle \{e, f\} \in B$.

3. $|B_1| = |B_2|$ for all $B_1, B_2 \in B$.
Delta-matroid

Definition

\((E, \mathcal{B})\) form a matroid if

1. \(\mathcal{B} \neq \emptyset\).
2. If \(B_1, B_2 \in \mathcal{B}\) and \(e \in B_1 \triangle B_2\) then there exists \(f \in B_1 \triangle B_2\) such that \(B_1 \triangle \{e, f\} \in \mathcal{B}\).
3. \(|B_1| = |B_2|\) for all \(B_1, B_2 \in \mathcal{B}\).

Definition (Bouchet)

\((E, \mathcal{F})\) form a \(\Delta\)-matroid if

1. \(\mathcal{F} \neq \emptyset\).
2. If \(F_1, F_2 \in \mathcal{F}\) and \(e \in F_1 \triangle F_2\) then there exists \(f \in F_1 \triangle F_2\) such that \(F_1 \triangle \{e, f\} \in \mathcal{F}\).

We allow \(e = f\) in the definition.

\(\mathcal{F}\) is the set of feasible sets.
Theorem (Bouchet)

- The quasi-trees of a ribbon graph form the feasible sets of a \( \Delta \)-matroid.
- The ribbon graph can be embedded in an orientable surface if and only if the sizes of all feasible sets have the same parity. (Such \( \Delta \)-matroids are called even).
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Proposition

Every matroid is a \( \Delta \)-matroid.
Theorem (Bouchet)

- The quasi-trees of a ribbon graph form the feasible sets of a $\Delta$-matroid.
- The ribbon graph can be embedded in an orientable surface if and only if the sizes of all feasible sets have the same parity. (Such $\Delta$-matroids are called even).

Proposition

Every matroid is a $\Delta$-matroid.

Proposition (Bouchet)

The feasible sets of smallest (largest) size form the bases of a matroid $D_{\text{min}}$ ($D_{\text{max}}$).
Deletion and contraction in ribbon graphs

<table>
<thead>
<tr>
<th></th>
<th>non-loop</th>
<th>non-orientable loop</th>
<th>orientable loop</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>G</strong></td>
<td><img src="G" alt="non-loop" /></td>
<td><img src="G-e" alt="non-orientable loop" /></td>
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</tbody>
</table>
Deletion and contraction in $\Delta$-matroids

Just like matroids . . .

Definition

A **coloop** is an element in every feasible set. A **loop** is an element in no feasible set.

Definition

1. If $e$ is not a coloop in $D$, then
   \[ \mathcal{F}(D \setminus e) = \{ F : F \in \mathcal{F}(D), e \not\in F \}. \]

2. If $e$ is not a loop in $D$, then
   \[ \mathcal{F}(D/e) = \{ F - e : F \in \mathcal{F}(D), e \in F \}. \]

3. If $e$ is either a coloop or a loop then $D/e = D \setminus e$. 
Deletion and contraction in ribbon graphs corresponds to deletion and contraction in Δ-matroids.

\[ D(G) \quad D(G/1) \quad D(G \setminus 1) \]

<table>
<thead>
<tr>
<th>123</th>
<th>23</th>
<th>2 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>2</td>
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</table>
The partial dual with respect to an arbitrary set $A$ is formed by taking the partial dual with respect to the edges of $A$ one edge at a time.
Partial duals...

G= has four partial duals (up to isomorphism):

Observe that $G$ and $G^A$ can have very different graph theoretic and topological properties.
Partial dual

Definition (Bouchet)

The partial dual or twist of the $\Delta$-matroid $D = (E, \mathcal{F})$ by a set $A \subseteq E$ is the $\Delta$-matroid $D \star A$ on $E$ with feasible sets

$$\{ F \triangle A : F \in \mathcal{F} \}.$$ 

Proposition (CMNR)

Partial duals of ribbon graphs and $\Delta$-matroids correspond.
A natural ribbon graph operation is the Partial Petrial: apply a half-twist to an edge \( e \) to get \( G^{\tau(e)} \).

Extends to sets of edges: \( G^{\tau(A)} \) denotes \( G \) with half-twists added to the edges of \( A \).
Minors and the medial graph

Any ribbon graph has an associated medial graph, which is a 4-regular graph $M(G)$ with a vertex corresponding to each edge of $G$ and edges corresponding to consecutive edges of $G$ in boundary walks.

![Diagram showing the medial graph transformation for different operations on $G$]

<table>
<thead>
<tr>
<th>Operation</th>
<th>$G$</th>
<th>$M(G)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G$</td>
<td><img src="image" alt="Graph $G$" /></td>
<td><img src="image" alt="Medial graph $M(G)$" /></td>
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Penrose polynomial for plane / embedded graphs

For a ribbon graph $G$, 

$$P(G; \lambda) = \sum_{A \subseteq E} (-1)^{|A|} \lambda^{f(G'^{(A)})},$$

where $f(G)$ is the number of boundary components of $G$. 

Theorem (Aigner, Ellis-Monaghan, Moffatt) 

$$P(G; \lambda) = P(G/e; \lambda) P(G'^{(e)}; \lambda)$$

for any edge $e$ of a ribbon graph $G$. If $G$ has no edges, 

$$P(G; \lambda) = \lambda^{f(G)}.$$
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$$P(G; \lambda) = P(G/e; \lambda) - P(G^{\tau(e)}/e; \lambda)$$

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Theorem (Aigner, Ellis-Monaghan, Moffatt)

For any ribbon graph $G$,

$$P(G; \lambda) = \sum_{A \subseteq E} (-1)^{|A|} \chi((G^\tau(A))^*; \lambda).$$
Binary delta-matroids

A symmetric binary matrix

\[
\begin{bmatrix}
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 \\
\end{bmatrix}
\]

The non-singular principal sub-matrices

\[
\begin{array}{cccc}
1234 \\
124 & 134 \\
12 & 13 & 34 \\
4 & & \\
\emptyset & & & \\
\end{array}
\]
Binary delta-matroids

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\]

Definition (Bouchet)

Binary $\Delta$-matroids are partial duals of those formed from symmetric binary matrices.
A symmetric binary matrix

\[
\begin{bmatrix}
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1
\end{bmatrix}
\]

The non-singular principal sub-matrices

\[
\begin{array}{cccc}
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12 & 13 & 4 & \emptyset
\end{array}
\]

Proposition (Bouchet)

*Every binary matroid is a binary \( \triangle \)-matroid.*
Loop complementation

If $M$ is a symmetric binary matrix then $M + e$ is formed from $M$ by replacing $M_{e,e}$ by $1 - M_{e,e}$.

<table>
<thead>
<tr>
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This can be extended to $M + A$ in the obvious way.

**Proposition (Brijder, Hoogeboom)**

1. If $e \notin F$ then $F \in \mathcal{F}(D(M + e)) \Leftrightarrow F \in \mathcal{F}(D(M))$
2. If $e \in F$ then $F \in \mathcal{F}(D(M + e)) \Leftrightarrow F - e \in \mathcal{F}(D(M))$ xor $F \in \mathcal{F}(D(M))$.

More generally this operation does not give a $\Delta$-matroid. A $\Delta$-matroid closed under partial duals and loop complementation is called $\text{vf-safe}$. 
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\begin{array}{c|c}
1 & v \\
\hline
v^t & A \\
\end{array}
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More generally this operation does not give a $\Delta$-matroid. A $\Delta$-matroid closed under partial duals and loop complementation is called \textit{vf-safe}.
Let $C$ and $C^*$ denote the cycle and cocycle spaces of a binary matroid respectively. Let

$$B_M(A) = \{ C \in C : C \cap A \in C^* \}.$$

**Definition (Aigner)**

For a binary matroid $M$,

$$P(M; \lambda) = \sum_{A \subseteq E} (-1)^{|A|} \lambda^{\dim(B_M(A))}.$$
Penrose polynomial for $\Delta$-matroids

**Definition (Brijder, Hoogeboom)**

For a vf-safe $\Delta$-matroid $D$,

$$P(D; \lambda) = \sum_{A \subseteq E} (-1)^{|A|} \lambda^{f(D+A)},$$

where $f(D) = |E(D)| - \max\{|F| : F \in \mathcal{F}(D)\}$. 
### Proposition (Brijder, Hoogeboom)

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1. If $e \not\in F$ then $F \in \mathcal{F}(D(G^{\tau(e)})) \iff F \in \mathcal{F}(D(G))$  
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Tying things up: 2

\[ f(D) = |E(D)| - \max\{|F| : F \in \mathcal{F}(D)\}. \]
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\[ f(G) \text{ is the number of boundary components of } G. \]
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**Proposition (CMNR)**

*For a ribbon graph \( G \),*

\[ P(G; \lambda) = \lambda^{\kappa(G)} P(D(G); \lambda). \]
From ribbon graphs to $\Delta$-matroids

**Theorem (Aigner, Ellis-Monaghan, Moffatt)**

For any ribbon graph $G$,

$$P(G; \lambda) = \sum_{A \subseteq E} (-1)^{|A|} \chi((G^{\tau(A)})^*; \lambda).$$
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Theorem (CMNR)

For any vf-safe $\Delta$-matroid $D$,

$$P(D; \lambda) = \sum_{A \subseteq E} (-1)^{|A|} \chi(((D + A)^*)_{\text{min}}; \lambda).$$
Questions

Any questions?