A Tutte-like polynomial for polymatroids

Amanda Cameron
Queen Mary, University of London

Matroids & the Tutte polynomial

The Tutte polynomial, originally formulated for graphs, has been generalised to apply to matroids:

Definition

Let $M = (E, r)$ be a matroid. The **Tutte polynomial** of M is

$$T_M(x, y) = \sum_{S \subseteq E} (x-1)^{r(E)-r(S)}(y-1)^{|S|-r(S)}.$$

Example: $T_{M(K_3)} = x^2 + x + y$

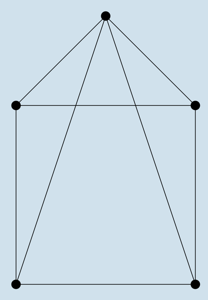
By substituting in for x and y, we get *Tutte invariants*. For instance:

$$T_M(2,1) = |\mathcal{I}(\mathcal{M})|$$
$$T_M(1,1) = |\mathcal{B}(\mathcal{M})|$$

We can view matroids geometrically as polytopes – there is a simple construction for this, as shown in the following example:

Example. Let $M = (E, \mathcal{B})$ be the matroid with $E = [4]$, $\mathcal{B} = \{12, 13, 14, 23, 24\}$.

$$\begin{aligned} P(M) &= \text{ConvHull}\{e_{\mathbf{B}} \mid B \in \mathcal{B}\} \\ &= \text{ConvHull}\{1100, 1010, 1001, 0110, 0101\} \\ &= \end{aligned}$$



Polymatroids

We now consider an extension of matroids called *polymatroids*. These are geometric objects which are formed by relaxing one rank condition of matroids.

Definition

A **polymatroid** consists of a finite ground set E and a rank function $r : 2^E \to \mathbb{N}$ which satisfies R2, R3 and has $r(\emptyset) = 0$.

Example: Take a graph $G = (V, E)$. We can construct a polymatroid $PM(G) = (E, r)$ by taking $E = E(G)$ and define the rank $r(X)$ of a subset $X \subset E$ to be the number of vertices incident with V. For instance, if we take $G = K_3$ again, we have that $r(\{a\}) = 2$ and $r(\{a, b\}) = 3$.

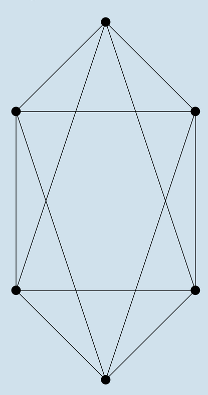
To see polymatroids geometrically, we form the vertices of a polytope by using the Greedy Algorithm. Choose some ordering of the ground set E, and let $S_i = \{1, \dots, i\}$ according to the chosen ordering. Form a vector $\mathbf{x} = (x_1, \dots, x_{|E|})$ by:

$$x_1 = r(S_1)$$
$$x_i = r(S_i) - r(S_{i-1})$$

The vertices of a polymatroid are these vectors $\mathbf{x}$ formed for all possible orderings of the ground set.

Example. Take the polymatroid of K_3 as defined above.

$$P(M) = \text{ConvHull}\{210, 201, 120, 021, 102, 012\}$$



Our Goal

The Tutte polynomial doesn't apply to polymatroids – for instance, one can define minors of (poly)matroids analoguously to graph minors. The Tutte polynomial has formulae involving minors which do not apply to polymatroids.

Aim

We will construct a two-variable polynomial which "acts" like Tutte in some way for polymatroids, and is equivalent to Tutte for matroids.

Note that the number of bases in M is the number of lattice points in $P(M)$. We will thus form a polynomial which counts the number of lattice points in a particular polytope which we construct from $P(M)$ in a way which introduces two variables.

Construction of a new polynomial

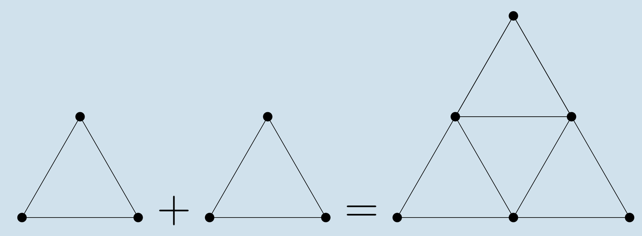
Definition

The **Minkowski sum** of two polytopes P and Q in $\mathbb{R}^n$ is

$$P + Q = \{p + q \mid p \in P,\ q \in Q\},$$

which is also a polytope.

Example:



This operation clearly results in a polytope with a greater number of lattice points.

We construct $\mathbf{P(M)} + \mathbf{u}\mathbf{\Delta}^{|\mathbf{E}|-\mathbf{1}} + \mathbf{t}(-\mathbf{\Delta}^{|\mathbf{E}|-\mathbf{1}})$ for $u, t \in \mathbb{N}$ where $\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^n \mid \sum_{i=0}^n t_i = 1 \text{ and } t_i \geq 0\ \forall i\}$. Then we count lattice points of this polytope:

$$\begin{aligned} Q(u,t) :&= \#((P(M) + u\Delta + t(-\Delta)) \cap \mathbb{Z}^{|E|}) \\ &= \sum_{i,j} c_{ij} \binom{u}{j}\binom{t}{i} \end{aligned}$$

To get a candidate for our Tutte-like polynomial for polymatroids, we perform a change of basis, and end up with:

Definition [Polymatroid Laurent polynomial]

$$Q'(M; x, y) := \sum_{i,j} c_{ij}(x-1)^i(y-1)^j$$

where the c_{ij} are those from $Q(x, y)$. When M is a (poly)matroid, this is a (Laurent) polynomial.

Example: $Q'(M(K_3); x, y) = x^2 + 2xy + y^2 - x$

Justification

This polynomial turns out to be a good candidate: when M is a matroid but not a polymatroid, our polynomial is an evaluation of the Tutte polynomial:

Theorem

Let M be a matroid.

$$T(x,y) = -\frac{(xy-x-y)^{|E|-1}}{(-y)^{r-1}(-x)^{|E|-r-1}} Q'\left(\frac{-y}{xy-x-y}, \frac{-x}{xy-x-y}\right)$$

Alternatively,

$$Q'(M; x, y) = \frac{x^r y^{|E|-r}}{x+y-1} T_M\left(\frac{x+y-1}{x}, \frac{x+y-1}{y}\right)$$

We are now investigating how properties of Q' for polymatroids relate to properties of T for matroids and graphs, with the hopes of claiming Q' as the natural Tutte-like polynomial for polymatroids. This includes providing interpretations of the coefficients of Q'. An important tool for this is subdivisions:

Definition

Let P be a polytope. A **subdivision** of P is a set of polytopes $P_1, \dots, P_m$, such that $P_1 \cup P_2 \cup \dots \cup P_m = P$, and, for all i, j, if P_i and P_j intersect, the intersection is a face of both P_i and P_j. A subdivision is **matroidal** if each P_i, and P, is a matroid.

One expected upcoming result on the coefficients of Q' is the following:

Expected Proposition

Let M be a matroid. Then $|[x^i y^j]Q'(M; x, y)|$ counts the cells $F + G + H$ of the appropriate mixed subdivision of $\Delta + P(M) + \nabla$, where G is a vertex of $P(M)$ and there exists no cell $F + G' + H$ such that $G' \not\supseteq G$, and $i = dim F, j = dim H$.