PROBLEMS FROM THE WORKSHOP ON THE TUTTE POLYNOMIAL ROYAL HOLLOWAY UNIVERSITY OF LONDON 11-14TH JULY 2015

Problems posed by Peter Cameron

$\mathcal{G}$-invariant and trellis decoding

Trellis decoding is a method of error correction which takes account of the fact that in a noisy channel, the received word is a vector of reals rather than a word over the finite alphabet of the code; it finds the codeword at shortest Euclidean distance from the received word. (Assuming that errors are independent and Gaussian, this coincides with maximum likelihood decoding.)

A *trellis* is a directed graph with a source and sink, having edges labelled with the elements of the alphabet of the code, so that the words given by reading paths from source to sink coincide with the codewords. As each value of the incoming word is received, the contribution to the Euclidean distance for each edge can be computed, and when the whole word is received, we just have a shortest path problem to solve.

Muder [2] showed how to construct a trellis for a linear code over a finite field, which is minimal in several senses (number of vertices or of edges, or cycle rank). The size of the trellis depends on the positions of elements of the (lexicographically) first and last bases of the associated matroid. The smallest trellis is obtained as a compromise from making the first basis occur as late as possible and making the last basis occur as early as possible.

For example, for the extended binary Golay code, the smallest trellis has 2686 vertices, with last basis $\{8, 12, 14, 15, 16, 18, 19, 20, 21, 22, 23, 24\}$, and the first basis the complement of this. The orderings giving such a trellis can be characterised in terms of the geometry of the code.

The *$\mathcal{G}$-invariant* of a matroid [1] is a gadget which keeps track of the positions of elements of the first base of the matroid under all possible orderings of its elements.

Problem. Is there an analogue of the $\mathcal{G}$-invariant which keeps track of the first and last bases under all possible orderings? How is such an invariant related to the Tutte polynomial? Is it possible to obtain some kind of "average" size of the trellis of a linear code as a specialisation of the Tutte polynomial of the associated matroid?

References

[1] H. Derksen, Symmetric and quasi-symmetric functions associated to polymatroids, *J. Algebr. Comb.* **30** (2009), 43–86.

[2] D. J. Muder, Minimal trellises for block codes, *IEEE Trans. Inform. Theory* **IT-15** (1969), 177–179.

Problems posed by Fengming Dong

$P(G, x)$: the chromatic polynomial of G

$F(G, x)$: the flow polynomial of G

Conjecture 0.1. *For any graph G, if $P(G, x) = 0$ and $x \neq 0$, then $Re(x) \neq 0$.*

Conjecture 0.2. *There is a constant μ such that $F(G,x) > 0$ holds for any bridgeless graph G and any real $x \geq \mu \cdot g(G)$, where $g(G)$ is the girth of G.*

Conjecture 0.3. *There exists a sequence $c_2, c_3, \cdots$ such that*

$$F(G,x) > 0$$

for all graphs G with $g(G) = i$ and real $x \geq c_i$, and

$$\lim_{n\to\infty} c_n/n = 0.$$

Conjecture 0.4. *For any bridgeless graph G, if all zeros of $F(G,x)$ are real, then all zeros of $F(G,x)$ are in $\{1,2,3,4\}$.*

Conjecture 0.5. *For any Hamiltonian graph G, $P(G,x)$ has no zeros in the interval $(1,2)$.*

Problems posed by Alex Fink

This problem concerns the generalisation of Fink and Speyer's construction of the Tutte polynomial to delta-matroids.

Delta-matroids are type BC Coxeter matroids for the maximal parabolic subgroup excluding only the simple root α_n of unusual length (the one long in type C, and short in type B).

If M is a representable delta-matroid, represented by the point x in the flag variety $Fl(\alpha_n)$, then the moment polytope of the closure $\overline{xT}$ of its orbit under the usual action of an n-dimensional algebraic torus T is the convex hull of the bases of M. Its class $[M] = [\mathcal{O}_{\overline{xT}}]$ in the T-equivariant algebraic K-theory of $Fl(\alpha_n)$ is determined locally by the vertex figures of the polytope; this allows a class $[M]$ to be defined even when M is not representable.

Let $\mathcal{O}(1)$ be the ample generator of $\mathrm{Pic}(Fl(\alpha_n))$. Let A be some collection of other simple roots: plausible choices are a singleton set $\{\alpha_1\}$ or $\{\alpha_2\}$ (in Bourbaki's labelling). Define the double fibration

$$Fl(\alpha_n) \xleftarrow{\ \pi\ } Fl(A \cup \{\alpha_n\}) \xrightarrow{\ \rho\ } Fl(A).$$

One can then define an invariant of M as

$$\rho_* \pi^*([M] \cdot [\mathcal{O}(1)]) \in K_T^0(Fl(A)).$$

How do the invariants defined this way compare to known Tutte-like invariants of delta-matroids, such as the Bollobas-Riordan polynomial?

Problems posed by Andrew Goodall

Homomorphism counting and matroid invariants

A group Γ acting on a set V ($\gamma \in \Gamma$ sending $v \in V$ to $v\gamma \in V$) is *generously transitive* if for each $u, v \in V$ there is γ such that $u\gamma = v$ and $v\gamma = u$. Let G, H be finite simple graphs and let $\hom(G,H)$ denote the number of homomorphisms from G to H.

(i) Is it true that if the graph invariant

$$h(G) = \frac{\hom(G,H)}{|V(H)|^{c(G)}}$$

depends only on the cycle matroid of G then the automorphism group of H must be generously transitive? (The converse holds [de la Harpe and Jaeger, 1995].)

(ii) Suppose that $h(G)$ is indeed dependent only on the cycle matroid of G, i.e.,

$$h(G) = f(M(G))$$

where $M(G)$ is the cycle matroid of G. What does $f(M(G))$ count in terms of the matroid $M(G)$? For which matroids other than graphic matroids can it be defined?

(iii) Under the hypotheses of (ii), what are the matroid duals of the objects counted by $f(M(G))$ expressed in terms of the graph G?

(iv) Is there a weighted graph $\widehat{H}$ such that $f(M(G)^*) = \hom(G, \widehat{H})$?

The motivating example for (ii)-(iv) is when $H = K_k$, where $f(M(G))$ counts nowhere-zero $\mathbb{Z}_k$-tensions of $M(G)$ with an orientation (a signing of the circuits encoding an orientation of the graph G). Tensions can be defined for any orientable matroid (circuits signed subject to a consistency condition). Nowhere-zero $\mathbb{Z}_k$-flows are dual to nowhere-zero $\mathbb{Z}_k$-tensions, and flows can be defined by stipulating that Kirchhoff's law holds at each vertex of G. The graph $\widehat{K_k}$ with weight -1 on the edges of the complete graph K_k, loops of weight $k-1$ on each vertex, and a weight of $\frac{1}{k}$ on each vertex has the property that $\hom(G, \widehat{H}) = F(G; k)$, the flow polynomial of G.

Cayley graphs have generously transitive automorphism group. Kneser graphs and generalized Johnson graphs also have generously transitive automorphism group, but in general are not Cayley graphs. Thus for example "Kneser $k : r$ colourings" of G (homomorphisms to $K_{k:r}$) depend only on the cycle matroid of G and the number of connected components of G: the question (iii) in this case is what might count as"Kneser $k : r$ flows". (When $r = 1$ these are nowhere-zero $\mathbb{Z}_k$-flows.)

Problems posed by Jesper Jacobsen

to follow

Problems posed by J.A. Makowsky

Let P be a graph property. A vertex P-coloring with at most k colors of a graph $(V(G), E(G))$ is a function

$$f : V(G) \to [k]$$

such that for each $i \in [k]$ the set $f^{-1}(i)$ induces a subgraph of G which is in P.

An edge P-coloring with at most k colors of a graph $(V(G), E(G))$ is a function

$$f : E(G) \to [k]$$

such that for each $i \in [k]$ the spanning subgraph of the set $f^{-1}(i)$ is a subgraph of G which is in P.

In extremal graph theory many such colorings are studied, usually motivated by concepts in engineering are natural sciences.

Theorem (Kotek,Makowsky Zilber):
Let $\chi_v^P(G; k)$ be the number of vertex P-colorings with at most k colors of a graph G (and $\chi_e^P(G; k)$ be the analogue for edge colorings). Then for each G the number $\chi_v^P(G; k)$ ($\chi_e^P(G; k)$) is a polynomial in k.

Theorem (Makowsky,Ravve): For two graph properties P, Q the polynomials $\chi_v^P(G; k)$ and $\chi_v^Q(G; k)$ ($\chi_e^P(G; k)$ and $\chi_e^Q(G; k)$) are d.p. equivalent iff $P = Q$ or $P = \neg Q$.

Problem 1: Study these polynomials! Location of zeroes, expressive power, etc.

Let $P(G,A)$ be property of subsets $A \subseteq V(G)^r$ of a graph G. Let $f_1(G), \ldots, f_m(G)$ be integer-valued graph parameters. We look at the graph polynomials of the form

$$F_{P,f_1,\ldots,f_m}(G, X_1, \ldots, X_m) = \sum_{A \subseteq V(G)^r : P(G,A)} \left(\prod_{i=1}^{m} X^{f_i(G[A])} \right)$$

This clearly generalizes the Tutte polynomial.

Problem 2: Study these polynomials! For which properties and graph parameters is it interesting.

Averbouch, Godlin and Makowsky have shown that there is such a polynomial generalizing both the Tutte and the matching polynomial, which is characterized by deletion, contraction and extraction of edges.

An extension of the bivariate chromatic polynomial I Averbouch, B Godlin, JA Makowsky European Journal of Combinatorics 31 (1), 1-17

Problem 3: How are the generalized chromatic polynomials $\chi_v^P(G;k)$ and the polynomials $F_{P,f_1,\ldots,f_m}(G, X_1, \ldots, X_m)$ related?

Problems posed by Iain Moffatt

A *delta-matroid* $D = (E, \mathcal{F})$ consists of a set E and a non-empty collection $\mathcal{F}$ subsets of E that satisfies the *Symmetric Exchange Axiom*: for all $X, Y \in \mathcal{F}$, if there is an element $u \in X \triangle Y$, then there is an element $v \in X \triangle Y$ such that $X \triangle \{u, v\} \in \mathcal{F}$. Elements of $\mathcal{F}$ called *feasible sets* and E is the *ground set*. For sets X and Y, $X \triangle Y := (X \cup Y) \backslash (X \cap Y)$ is their *symmetric difference*.

A delta matroid is a *matroid* if all of the feasible sets of a delta-matroid are equicardinal. This is the bases definition of a matroid. A matroid can also be defined in terms of a rank function $r : E \to \mathbb{N}_0$.

Question: Is there a "rank function" definition of a delta-matroid?

Perhaps the following will provide some insight as to what such a function should look like. Recall that the Tutte polynomial of a matroid M with rank function r is

$$T_M(x, y) = \sum_{A \subseteq E} (x-1)^{r(G)-r(A)} (y-1)^{|A|-r(A)}.$$

Restricting the feasible sets of a delta-matroid to those of maximal (respectively, minimal) size results in a matroid $D_{\max}$ (respectively, $D_{\min}$). The "Tutte polynomial" of a delta-matroid is

$$\tilde{R}_D(x, y) := \sum_{A \subseteq E} (x-1)^{\rho(E)-\rho(A)} (y-1)^{|A|-\rho(A)}.$$

where

$$\rho(D) := \frac{1}{2}(r_{\max}(D) + r_{\min}(D)),$$

where $r_{\max}(D)$ and $r_{\min}$ are the rank functions of $D_{\max}$ and $D_{\min}$ respectively. For $A \subseteq E$,

$$\rho(A) := \rho(D \backslash A^c).$$

It is important to notice that in general $\rho_D(A) \neq \frac{1}{2}(r_{D_{\max}}(A) + r_{D_{\min}}(A))$.

Question: Is there a way to determine the feasible sets of D from a knowledge of ρ? *(Update 16th July 2015: Carolyn Chun and Steve Noble showed that the answer is yes.)*
