Delta-matroid polynomials and the symmetric Tutte polynomial

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Motivation

Definition (Aigner & van der Holst 2004, and Bouchet 1991)

The (single-variable) *interlace polynomial* of a graph G is

$$q(G;y) = \sum_{X \subseteq V(G)} (y-1)^{n(A(G)[X])}.$$

Theorem (Aigner & van der Holst 2004, and Bouchet 1991)

Let M be a binary matroid and G be the fundamental bipartite graph of M with respect to some basis B. Then $T(M;y,y) = q(G;y)$.

- $T(M;y,y)$ is defined for arbitrary matroids (instead of only binary matroids).
- $q(G;y)$ is defined for arbitrary graphs (instead of only bipartite graphs).
- Goal: common generalization for $T(M;y,y)$ and $q(G;y)$.

Summary

	2 directions (2 minors ops)	3 directions (3 minors ops)
2-in,out/4-regular graphs G	Martin $m(G;y)$	Martin $M(G;y)$
looped simple graphs G	Interlace $q(G;y)$	Interlace $Q(G;y)$
matroids	Tutte $T(M;y,y)$	—
Δ-matroids D	$q(D;y)$	$Q(D;y)$

Δ-matroids [Bouchet 1988] generalize both adjacency matrices and matroids.

Key: $q(D;y)$ and $Q(D;y)$ retain many of the attractive properties: recursive relations, various evaluations, etc.

Δ-matroids

Let Δ be symmetric difference.

Definition (Bouchet 1988)

A nonempty set system $D = (V, B)$ is a Δ-*matroid* over V if for all $X, Y \in B$ and $u \in X \,\Delta\, Y$, there is an element $v \in X \,\Delta\, Y$ such that $X \,\Delta \{u, v\} \in B$ (we allow $u = v$).

Theorem (Bouchet 1988)

A set system is a matroid (described by its bases) iff it is an equicardinal Δ-matroid.

Δ-matroids

Definition

Twist of D on $X \subseteq V$ is $D * X := (V, B * X)$ where $B * X = \{Y \Delta X \mid Y \in D\}$.

Twist generalizes matroid duality: $M * V = M^*$.

Theorem (Bouchet 1988)

D is a Δ-matroid iff $D * X$ is a Δ-matroid.

Δ-matroids have deletion and contraction, generalizing deletion and contraction for matroids.

Theorem

A set system D is a Δ-matroid iff $\min(D * X)$ is equicardinal for all $X \subseteq V$.

Representable Δ-matroids

Theorem (Bouchet 1988)

For a skew-symmetric $V \times V$-matrix A (over a field $\mathbb{F}$), $\mathcal{D}_A := (V, B_A)$ with $B_A = \{X \subseteq V \mid A[X]$ nonsingular$\}$ is a Δ-matroid.

Definition

A Δ-matroid D over V is *representable over* $\mathbb{F}$ if $D = \mathcal{D}_A * X$ for a $V \times V$-skew-symmetric A over $\mathbb{F}$ with $X \subseteq V$.

Theorem (Bouchet 1988)

A matroid is representable over $\mathbb{F}$ in the usual matroid sense iff it is representable over $\mathbb{F}$ in this Δ-matroid sense.

A Δ-matroid D is called *binary* if representable over $GF(2)$.

interlace polynomial as Δ-matroid polynomial

Definition

Let $D = (V, B)$ be a Δ-matroid. Define d_D as the common cardinality of the elements of $\min(B)$.

So, d_{D*X} is Hamming distance of X from D.

Theorem

For any graph G, $d_{\mathcal{D}_{A(G)}*X} = n(A(G)[X])$.

Corollary

For any graph G,

$$q(G; y) = \sum_{X \subseteq V(G)} (y-1)^{n(A(G)[X])} = \sum_{X \subseteq V(G)} (y-1)^{d_{\mathcal{D}_{A(G)}*X}}.$$

interlace polynomial as Δ-matroid polynomial

Corollary

For any graph G,

$$q(G;y) = \sum_{X \subseteq V(G)} (y-1)^{d_{\mathcal{D}_{A(G)} * X}}.$$

Generalization from binary Δ-matroids to arbitrary Δ-matroids:

Definition (Δ-matroid polynomial)

Let D be a Δ-matroid over V.

$$q(D;y) := \sum_{X \subseteq V} (y-1)^{d_{D*X}}.$$

So, $q(G;y) = q(\mathcal{D}_{A(G)};y)$.

symmetric Tutte polynomial as Δ-matroid polynomial

Definition (Tutte polynomial)

Let M be a matroid over V.

$$T(M; x, y) := \sum_{X \subseteq V} (x-1)^{n_{M^*}(V \setminus X)} (y-1)^{n_M(X)}.$$

Recall that $n_M(X) = |X| - r_M(X)$.

$n_{M^*}(V \setminus X) + n_M(X) = d_{M*X}$

Theorem

Let M be a matroid over V

$$T(M; y, y) = \sum_{X \subseteq V} (y-1)^{d_{M*X}} = q(M; y).$$

Δ-matroid notions

Loop and coloop compatible with matroids.

Definition

Let $D = (V, B)$ be a Δ-matroid. $v \in V$ is

- *loop* if for all $X \in B$, $v \notin X$,
- *coloop* if $D * v$ is loop,
- *singular* if v is either loop or coloop.

Deletion and contraction compatible with matroids.

Definition (deletion)

Let $D = (V, B)$ be a Δ-matroid and $v \in V$. If v is not a coloop, then $D \setminus v := (V \setminus \{v\}, B')$ with $B' = \{X \in B \mid v \notin X\}$. If v is a coloop, then $D \setminus v := D * v \setminus v$.

Contraction: $D * v \setminus v$.

Recursive relation for Δ-matroid polynomial

Theorem

Let D be a Δ-matroid over V. If $V = \emptyset$, then $q(D; y) = 1$.
If $v \in V$ is nonsingular in D, then

$$q(D; y) = q(D \setminus v; y) + q(D * v \setminus v; y).$$

If $v \in V$ is singular in D, then

$$q(D; y) = yq(D \setminus v; y) = yq(D * v \setminus v; y).$$

Two types of minor operations: deletion and contractions.

For a graph G and $Y \subseteq V(G)$. Let $G + Y$ be the graph obtained from G by toggling the existence of loops for the vertices of Y.

Definition (Aigner & van der Holst 2004, and Bouchet 1991)

Let G be a graph. Then the *global interlace polynomial* of G is

$$Q(G; y) = \sum_{X \subseteq V(G)} \sum_{Y \subseteq X} (y-2)^{n(A(G+Y)[X])}.$$

Δ-matroid version

Definition

Let $D = (V, B)$ be a Δ-matroid (or, more generally, set system) and $X \subseteq V$. Define *loop complementation of D on X* by $D + X = (V, B')$ where $Y \in B'$ iff $|\{Z \in B \mid Y \setminus X \subseteq Z \subseteq Y\}|$ is odd.

$D + X$ not necessarily a Δ-matroid.

Theorem

Let A be a symmetric $V \times V$-matrix and $X \subseteq V$. Then $\mathcal{D}_{A+X} = \mathcal{D}_A + X$.

The class of binary Δ-matroids is closed under $+$. Extendable to $GF(4)$.

Δ-matroid version

Theorem

Let D be a Δ-matroid (or, more generally, set system). Then $(D + X) + X = D$. In fact, $+X$ and $*X$ are involutions that generate S_3 and commutes on disjoint sets.

Third involution: $D \bar{*} X := D + X * X + X = D * X + X * X$.

Δ-matroid version

Let $\mathcal{P}_3(V)$ be the set of ordered 3-partitions of V.

Definition

Let D be a Δ-matroid. Define

$$Q(D;y) = \sum_{(A,B,C)\in\mathcal{P}_3(V)} (y-2)^{d_{D*B\bar{*}C}}.$$

Theorem

Let G be a graph. Then $Q(G;y) = Q(\mathcal{D}_G;y)$.

Δ-matroid version

In general, a Δ-matroid D is *vf-safe* if applying any sequence of twist and loop complementation obtains a Δ-matroid.

$v \in V$ is *strongly nonsingular* if v is nonsingular and $D * v \neq D$.

Theorem

Let D be a vf-safe Δ-matroid and let $v \in V$.

1. *If v is strongly nonsingular in D, then*

$$Q(D; y) = Q(D \setminus v; y) + Q(D * v \setminus v; y) + Q(D \bar{*} v \setminus v; y).$$

2. *If v is not strongly nonsingular in D, then*

$$Q(D; y) = yQ(D \setminus v; y).$$

Three types of minor operations!

Evaluations

Theorem

Let D be a Δ-matroid.

1. *If D is even and $|V| > 0$, then $q(D; 0) = 0$.*
2. *If D is vf-safe, then $q(D; -1) = (-1)^{|V|}(-2)^{d_{D \bar{*} V}}$ (third direction!).*
3. *If D is vf-safe with $|V| > 0$, then $Q(D; 0) = 0$.*
4. *If D is binary, then $q(D)(3) = k\,|q(D)(-1)|$ for some odd integer k [Bouchet].*

Penrose polynomial

Definition

Let D be a vf-safe Δ-matroid. The Penrose polynomial of D is

$$P(D;y) = \sum_{X \subseteq V} (-1)^{|X|} y^{d_{D * V \bar{*} X}}.$$

Recursive relation is outside realm of matroids.

Theorem

Let D be a vf-safe Δ-matroid. If $V = \emptyset$, then $P_M(y) = 1$.
If $v \in V$ is

- *nonsingular in $D \bar{*} V$, then*
  $P(D;y) = P(D * v \setminus v; y) - P(D \bar{*} v \setminus v; y)$,
- *a coloop of $D \bar{*} V$, then $P(D;y) = (1-y)P(D * v \setminus v; y)$, and*
- *a loop of $D \bar{*} V$, then $P(D;y) = (y-1)P(D \bar{*} v \setminus v; y)$.*

Multivariate version to incorporate all these Δ-matroid polynomials.

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Thanks!