Tutte polynomial and orbit-counting

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New directions for the Tutte Polynomial
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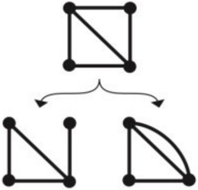

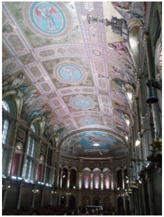

Much of this is joint work with
Dima Fon-Der-Flaass, Bill Jackson, Koko Kayibi, Jason Rudd

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In the second column, we count up to symmetry, specifically the symmetric group S_n, the automorphism group of the complete graph.

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The basic tool is the orbit-counting lemma:

Theorem

Let the finite group G act on the finite set X. Then the number of orbits of G on X is equal to the average number of fixed points of its elements:

$$\mathrm{Orb}(G, X) = \frac{1}{|G|} \sum_{g \in G} \mathrm{fix}(g, X).$$

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We would like to find polynomials for which suitable coefficients or evaluations give us the number of orbits of a group of automorphisms of the graph or matroid on the objects mentioned above (and others if possible).

Orbital chromatic polynomial

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Specifically,

$$\mathrm{OP}_{X,G}(q) = \frac{1}{|G|} \sum_{g \in G} P_{X/g}(q),$$

where P_X denotes the chromatic polynomial of X, and X/g denotes the graph obtained from X by shrinking each cycle of the permutation g to a single vertex (with a loop at that vertex if the cycle contains an edge of the graph – in this case $P_{X/g} = 0$).

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If the resulting polynomial is $\mathrm{OP}^*_{X,G}$, then the number of orbits on acyclic orientations is $(-1)^{|VX|}\,\mathrm{OP}^*_{X,G}(-1)$.

Flows and tensions

It is often said that flows are dual to colourings. Actually they are dual to tensions, and the orbital perspective makes this clear.

Let A be a finite abelian group. Take a fixed but arbitrary orientation of the edges of the graph X. An A-flow on X is a function from oriented edges to A such that the algebraic sum of the flows into any vertex is 0 (where flows out have a $-$ sign).

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A flow or tension is nowhere-zero if it does not take the value 0.

It is easy to see that the number of nowhere-zero tensions is $P_X(q)/q^\kappa$, where $q = |A|$, and P_X and κ are the chromatic polynomial and the number of connected components of X.

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Theorem

There are polynomials $\mathrm{OT}_{X,G}$ *and* $\mathrm{OF}_{X,G}$ *in indeterminates* $x_0, x_1, x_2, \ldots$ *such that the number of* G*-orbits on nowhere-zero* A*-tensions (resp.* A*-flows) are obtained from* $\mathrm{OT}_{X,G}$ *(resp.* $\mathrm{OF}_{X,G}$ *by substituting for* x_i *the number of solutions of* $ia = 0$ *in* A*.*

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Note that $x_0 = |A|$ and $x_1 = 1$. Moreover, x_i occurs in one of these polynomials only if G contains an element of order i. So we recover Tutte's observation when G is the trivial group.

Matrices over PIDs

The proof of the theorem involves finding a generalised Tutte polynomial associated with a matrix M over a principal ideal domain. The polynomial has two sequences of variables $x_0, x_1, x_2, \ldots$ and $y_0, y_1, y_2, \ldots$; the x variables give the orbital tension polynomial and the y variables the orbital flow polynomial.

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Now tensions are elements of the kernel of the vertex-edge incidence matrix of the graph, and flows are elements of the kernel of the cycle-edge incidence matrix. Both matrices are unimodular. To count fixed points of a permutation g with permutation matrix P_g, we append $I - P_g$ to the appropriate incidence matrix; this is no longer unimodular, so we need variables for all the possible invariant factors of the matrix (indexed by the associate classes in the ring $\mathbb{Z}$).

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All this can be done for matrices over any principal ideal domain.

Finite fields and codes

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So the general machinery gives us an *orbital weight enumerator* associated with a linear code and a group of automorphisms.
In the case of the trivial group, we get the weight enumerator of the code as a specialisation of our (four-variable) Tutte polynomial. Curiously, the connection we get is slightly different from the one in Curtis Greene's famous theorem!

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Let C be a linear code over $\mathbb{Z}_4$, the integers modulo 4. (These codes have been of some interest since the paper of Hammons *et al.* showed that certain famous non-linear binary codes such as the Nordstrom–Robinson, Preparata and Kerdock codes are images, under the non-linear Gray map, of linear $\mathbb{Z}_4$ codes.)

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The symmetrised weight enumerator of such a code has three indeterminates X_0, X_2, and X_{13}: each codeword w contributes a term $X_0^a X_2^b X_{13}^c$, where a, b and c are respectively the numbers of entries equal to 0, 2, and (1 or 3) respectively – in other words, the associate classes!

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So our approach gives a generalisation to codes over $\mathbb{Z}_q$, or more general finite rings.

Cycle index

Let G be a permutation group on a set X. The cycle index of G is the polynomial in indeterminates $s_1, s_2, \ldots$ given by

$$Z(G) = \frac{1}{|G|} \sum_{g \in G} s_1^{c_1(g)} s_2^{c_2(g)} \cdots ,$$

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This gadget automates many orbit counting problems. If we have a set of "figures" with non-negative integer "weights", with $A(x)$ the generating function of figures by weight, then any orbit of G on "functions" (from X to the set of figures) has an associated weight, and the generating function for functions by weight is given by the cycle index, with s_i substituted by $A(x^i)$ for all i: this is the cycle index theorem.

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If G is a group of automorphisms of a matroid M on X, can one combine the cycle index of G with the Tutte polynomial of M?

IBIS groups

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For IBIS groups, the connection between cycle index and Tutte polynomial should be closer.

Tutte cycle index

My first attempt went like this. Let G be an IBIS group, and M the associated matroid.

The Tutte cycle index of G is given by

$$\mathrm{ZT}(G) = \frac{1}{|G|} \sum_{A \subseteq X} u^{|G_A|} v^{b(G_{(A)})} Z(G_A^A),$$

where G_A and $G_{(A)}$ are the setwise and pointwise stabilisers of A, G_A^A the permutation group induced on A by G_A, and $b(G)$ is the base size of G.

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We recover the Tutte polynomial of M by the rule that $|G| ZT(G; u = 1, s_i = t^i)$ is equal to $t^{b(G)} T(M; v/t + 1, t + 1)$.

More generally?

For an arbitrary permutation group, the irredundant bases are not the bases of a matroid. Is there a more general combinatorial structure defined by these bases? Can we associate an analogue of the Tutte polynomial (or the Tutte cycle index) with it?

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Note that the first specialisation on the preceding slide works for an arbitrary permutation group; we could simply put $v = 1$ and omit all mention of matroid rank.