Some results and conjectures related to the critical problem

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The critical problem for matroids

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Provides a unified setting for such problems as

- Hadwiger's Conjecture
- Tutte's 5-Flow Conjecture

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The cycle matroid of every graph is representable over all fields.

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Let M be the matroid of the following matrix over $GF(2)$:

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \end{array}$$
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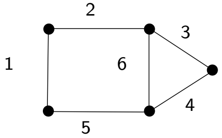
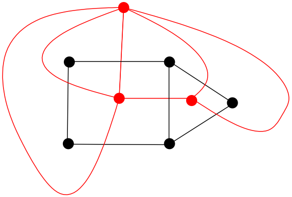
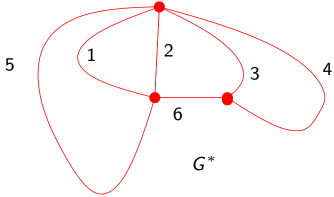


G





G^*

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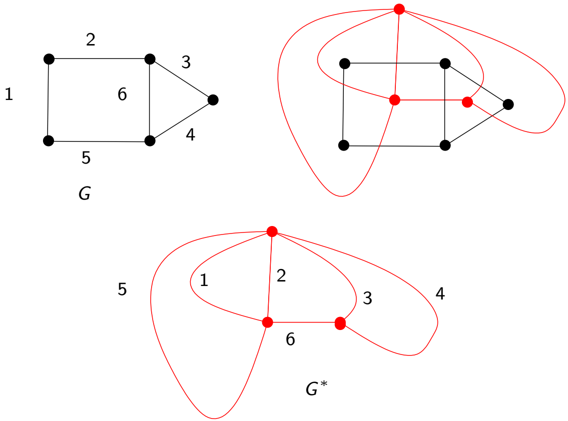

The **dual** $M^*(G)$ of $M(G)$ is $M(G^*)$.

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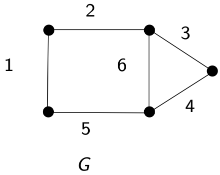

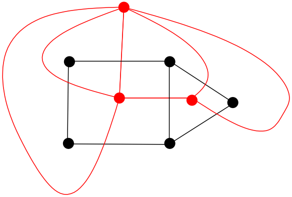

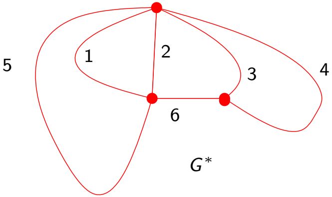

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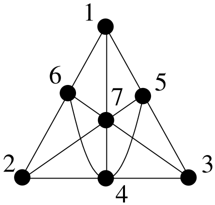

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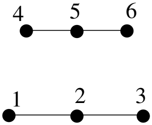

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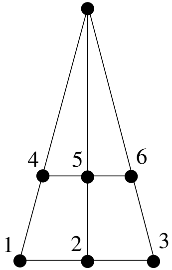

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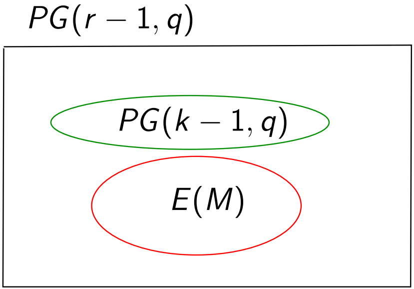

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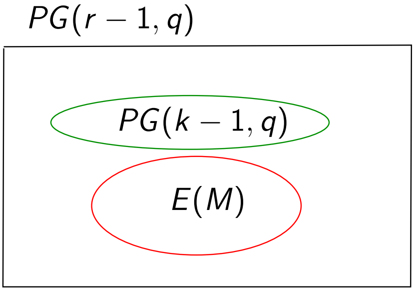

The critical exponent $c(M;q)$ of M is $r-k$.

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Theorem (Crapo and Rota, 1970)

If M has no loops, then $c(M;q) = \min\{j \in \mathbb{N} : p(M;q^j) > 0\}$.

If M has a loop, then $c(M;q) = \infty$.

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The following are equivalent for a loopless rank-r binary matroid M.

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(iv) *M^* can be partitioned into circuits.*

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Sharpened by Edwards (1973) in several ways; short proofs of these results given by Erdős, Gyárfás, and Kohayakawa (1997).

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A loopless non-empty $GF(q)$-representable matroid M has an affine restriction N such that

$$|E(N)| \geq \tfrac{q-1}{q}|E(M)| + \tfrac{1}{q}.$$

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The bound is sharp for the affine geometry $AG(r-1,q)$, which is $PG(r-1,q)\backslash PG(r-2,q)$.

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Corollary

A loopless non-empty binary matroid M has an affine restriction N that uses more than half the elements.

Nearly bipartite graphs

Theorem (Maffray, 1992)

A 2-connected simple graph G has no odd cycles of length exceeding three iff

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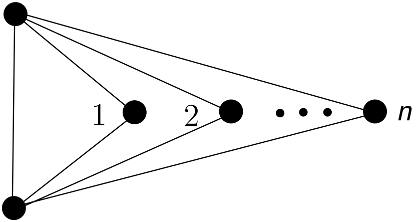

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A matroid is 2-connected if every two elements are contained in some circuit.

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Theorem (JGO, Wetzler, 2015)

A 2-connected simple binary matroid M has no odd circuits of size exceeding three iff

(i) *M is affine;*

(ii) *M is $M(K_4)$ or F_7; or*

(iii) *$M(G)$ is obtained by gluing together n triangles across a common element for some $n \geq 1$.*

Extensions

Theorem (Wetzler, 2015)

Let G be a 3-connected simple graph. If G has a 5-cycle but no larger odd cycle, then

(i) $|V(G)| \leq 6$; *or*

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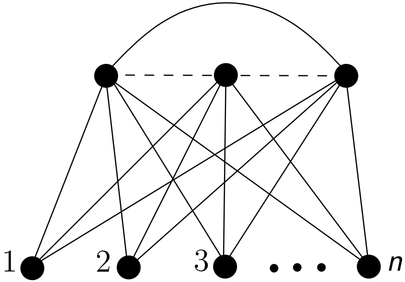

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Theorem (Wetzler, 2015)

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(i) $r(M) \leq 5$; *or*

(ii) *M is obtained by gluing together, across a common triangle, n matroids each isomorphic to $M(K_4)$ or F_7, and then possibly deleting one or two elements of the join triangle.*

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Conjecture (Tutte, 1966)

The only binary matroids with critical exponent exceeding 2 and all loopless proper minors having critical exponent at most 2 are

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Robertson, Sanders, Seymour, and Thomas have settled this conjecture for cubic graphs but it is still open in general.

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Conjecture (Brylawski, 1975)

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JGO (1987): True for $q = 3$.

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Theorem (Kung, 1987)

A loopless binary matroid with no $M(K_5)$-minor has critical exponent at most 8.

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This follows from Geelen, Gerards, and Whittle's announced result:

For all q, there are no infinite antichains of $GF(q)$-representable matroids.