Splitting Formulas for Tutte-Grothendiek Invariants on Graphs

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A **Tutte-Grothendieck (T-G) invariant** Φ is a mapping from a class of matroids to a commutative ring $(R, +, \cdot, 0, 1)$ such that for a matroid M on E we have **contruction-deletion rule**

$$
\begin{aligned}
\Phi(M) &= 1 && \text{if } E = \emptyset \\
\Phi(M) &= \alpha_1 \cdot \Phi(M - e) && \text{if } e \text{ is an isthmus of } M \\
\Phi(M) &= \beta_1 \cdot \Phi(M - e) && \text{if } e \text{ is a loop of } M \\
\Phi(M) &= \alpha_2 \cdot \Phi(M/e) + \beta_2 \cdot \Phi(M - e) && \text{otherwise.}
\end{aligned}
$$

Φ is determined by $(\alpha_1, \beta_1, \alpha_2, \beta_2), \quad \alpha_1, \beta_1, \alpha_2, \beta_2 \in R$

If $\alpha_1 = \alpha_2 + \beta_2$, then the 2nd row coincides with the 4th row and

Φ is called **isthmus-smooth**

If $\beta_2 \neq 0$ and $\xi \in R$ is a multiple of β_2, then $\Phi^{\mathrm{is}}_{\xi}(M) = \xi^{|E|} \left((\alpha_1 - \alpha_2)/\beta_2\right)^{r^*(M)} \Phi(M)$ is called a ξ-**isthmus-smooth modification** of $\Phi(M)$ and

$$\begin{aligned}
\Phi^{\mathrm{is}}_{\xi}(M) &= 1 && \text{if } E = \emptyset, \\
\Phi^{\mathrm{is}}_{\xi}(M) &= (\xi/\beta_2)\,\beta_1(\alpha_1 - \alpha_2)\,\Phi^{\mathrm{is}}_{\xi}(M - e) && \text{if } e \text{ is a loop of } G, \\
\Phi^{\mathrm{is}}_{\xi}(M) &= \xi\alpha_2\Phi^{\mathrm{is}}_{\xi}(M/e) + \xi(\alpha_1 - \alpha_2)\Phi^{\mathrm{is}}_{\xi}(M - e) && \text{otherwise.}
\end{aligned}$$

If $\beta_2 = 0$, then $\quad \Phi(M) = \beta_1^{r^*(M)} \alpha_1^{i_M} \alpha_2^{r(M)-i_M}$

where $i_M := \sharp$ isthmuses in M

Graph $G = (V, E)$, for $A \subseteq E$ define:

$c(A) = \sharp$ of components of the graph (V, A)

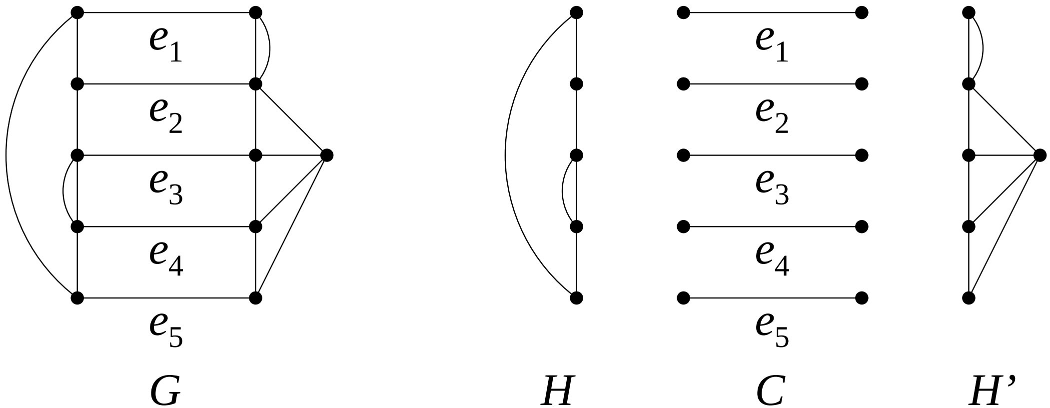
$r(A) = |V| - c(A)$ – rank of the matroid on G

$B_n = \{P_1, \ldots, P_{b_n}\}$ the set of partitions of $\{1, \ldots, n\}$

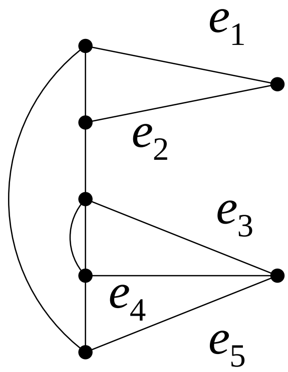
$|B_n| = b_n$ is the nth **Bell number**; $b_0 = 1$, $\forall n > 0$

$$b_n = \sum_{i=0}^{n-1} \binom{n-1}{i} b_i$$

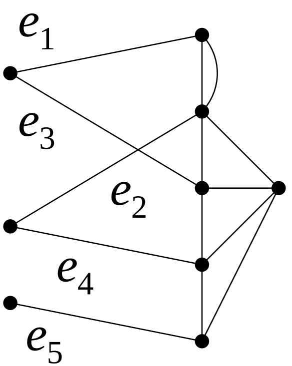
Let $C = \{e_1, \ldots, e_n\}$ be an n-edge-cut of G.



$$P = \big\{\{1,2\},\{3,4,5\}\big\} \in B_5 \qquad P' = \big\{\{1,3\},\{2,4\},\{5\}\big\} \in B_5$$

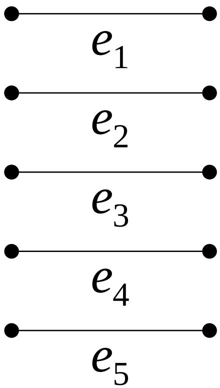


[H,P]

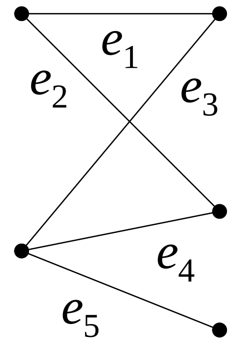


[H',P']

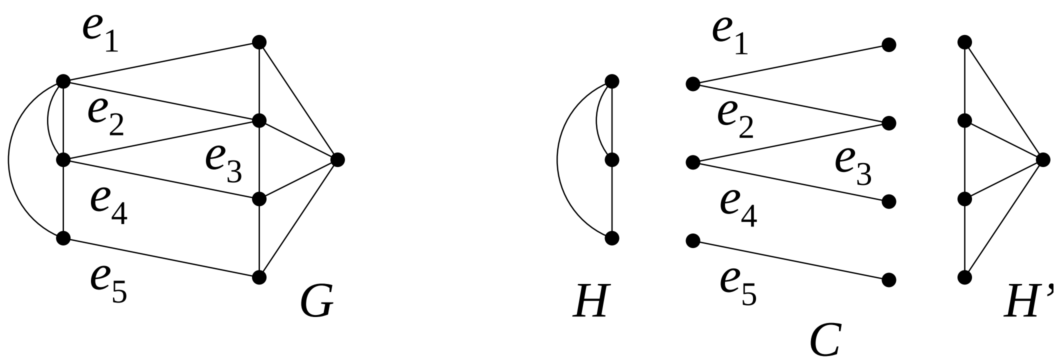
$P = \{\{1,2,\},\{3,4,5\}\}$



$P' = \{\{1,3\},\{2,4\},\{5\}\}$



[P,P’]



$$P_C = \{\{1,2,\},\{3,4\},\{5\}\} \qquad P'_C = \{\{1\},\{2,3\},\{4\},\{5\}\}$$

Let $P_C = \{Q_1, \ldots ,Q_p\}$

Define $B_C \subseteq B_n$ to be the set of partitions arising from P_C after unifying some sets from $\{Q_1, \ldots, Q_p\}$, $|B_C| = b_p$.

Similarly define B'_C from $P'_C = \{Q'_1, \ldots ,Q'_{p'}\}, \quad |B'_C| = b_{p'}$.

$M_n(\Phi)$: $b_n \times b_n$ matrix

$$\Big(M_n(\Phi)\Big)_{i,j} = \Phi([P_i, P_j]),\ 1 \le i, j \le n$$

$M_n^{H,H'}(\Phi)$: $(b_n{+}1) \times (b_n{+}1)$ matrix extending $M_n(\Phi)$

$$\Big(M_n^{H,H'}(\Phi)\Big)_{b_n+1,i} = \Phi([H, P_i]),\ 1 \le i \le n$$

$$\Big(M_n^{H,H'}(\Phi)\Big)_{i,b_n+1} = \Phi([H', P_i]),\ 1 \le i \le n$$

$$\Big(M_n^{H,H'}(\Phi)\Big)_{b_n+1,b_n+1} = \Phi(G)$$

Let $B, B' \subseteq B_n$

$M_{B,B'}(\Phi)$ is the $|B| \times |B'|$ submatrix of $M_n(\Phi)$ consisting of the rows (columns) corresponding to the partitions from B (B').

$M^{H,H'}_{B,B'}(\Phi)$ is the $(|B|+1) \times (|B'|+1)$ submatrix of $M^{H,H'}_n(\Phi)$ consisting of the rows (columns) corresponding to B (B').

Let $M^{H,H'}_{n,0}(\Phi)$ $\left(M^{H,H'}_{B,B',0}(\Phi)\right)$ arise from $M^{H,H'}_n(\Phi)$ $\left(M^{H,H'}_{B,B'}(\Phi)\right)$ after replacing $\Phi(G)$ by 0 in the entry of the last row and column.

Theorem 1: Let Φ be an isthmus-smooth T-G invariant, G be a graph with an n-edge cut C, $B_C \subseteq B \subseteq B_n$, $B'_C \subseteq B' \subseteq B_n$. Suppose that $\tilde{B} \subseteq B$, $\tilde{B}' \subseteq B'$ such that $M_{\tilde{B},\tilde{B}'}(\Phi)$ is a maximal regular submatrix of $M_{B,B'}(\Phi)$. Then

$$\left|M^{H,H'}_{\tilde{B},\tilde{B}'}(\Phi)\right| = 0, \text{ whence}$$

$$\Phi(G) = -\left|M^{H,H'}_{\tilde{B},\tilde{B}',0}(\Phi)\right| \cdot \left|M_{\tilde{B},\tilde{B}'}(\Phi)\right|^{-1}.$$

The **Tutte polynomial** of a matroid M with rank r is

$$T(M;x,y) = \sum_{A\subseteq E}(x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}$$

$T(M;x,y)$ is a T-G invariant on $\mathbb{Z}[x,y]$ determined by $(1,1,x,y)$

Let $T'(G;x,y)$ be the 1-isthmus-smooth modification of $T(G;x,y)$

$$T'(G;x,y) = T_1^{\mathrm{is}}(G;x,y) = (x-1)^{|E|-r(G)}T(G;x,y)$$

If $\Phi(M)$ is a T-G invariant determined by $(\alpha_1,\beta_1,\alpha_2,\beta_2)$, then

$$\Phi(M) = \alpha_2^{r(M)}\beta_2^{r^*(M)}T(M;\alpha_1/\alpha_2,\beta_1/\beta_2)$$

$M_n(T')$ and $M_{B,B}(T')$ are regular $\forall\, n \geq 1$ and $\forall\, B \subseteq B_n$

Theorem 2: Let C be an n-edge cut of a graph G. Then

$$\left|M_n^{H,H'}(T')\right| = 0, \text{ whence}$$

$$T'(G;x,y) \;=\; -\; \left|M_{n,0}^{H,H'}(T')\right| \,\cdot\, \left|M_n\;\;(T')\right|^{-1}.$$

Determinant formula of size $b_n + 1$.

$$B_2 = \Big\{ \big\{\{1\},\{2\}\big\},\ \big\{\{1,2\}\big\} \Big\}$$

$$M_2(T') \ = \ \begin{pmatrix} x^2 & x^2 \\ x^2 & (x+y)(x-1) \end{pmatrix}$$

$$B_3 =$$

$$\Big\{ \{\{1\},\{2\},\{3\}\}, \{\{1,2\},\{3\}\}, \{\{1\},\{2,3\}\}, \{\{1,3\},\{2\}\}, \{\{1,2,3\}\} \Big\}$$

$$M_3(T') \;=$$

$$\begin{pmatrix} x^3 & x^3 & x^3 & x^3 & x^3 \\ x^3 & x(x+y)(x-1) & x^3 & x^3 & x(x+y)(x-1) \\ x^3 & x^3 & x(x+y)(x-1) & x^3 & x(x+y)(x-1) \\ x^3 & x^3 & x^3 & x(x+y)(x-1) & x(x+y)(x-1) \\ x^3 & x(x+y)(x-1) & x(x+y)(x-1) & x(x+y)(x-1) & (x+y+y^2)(x-1)^2 \end{pmatrix}$$

Theorem 3: Let C be an n-edge cut of a graph G, $P \in B_n$ such that $P = \{Q'_1, \ldots, Q'_{p'}\} \leq P_C = \{Q'_1, \ldots, Q'_p\}$, $p' \geq p$. Let ι be a mapping from $\{1, \ldots, p\}$ to $\{1, \ldots, p'\}$ such that $Q'_{\iota(i)} \subseteq Q_i$ for $i = 1, \ldots, p$ and B_ι be the set of partitions from B_n arising from P after unifying some sets from $\{Q'_{\iota(1)}, \ldots, Q'_{\iota(p)}\}$. Then

$$\left|M^{H,H'}_{B_C,B_\iota}(T')\right| = \left|M^{H,H'}_{B_C,B_C}(T')\right| = 0, \text{ and}$$

$$T'(G;x,y) = -\left|M^{H,H'}_{B_C,B_\iota,0}(T')\right| \cdot \left|M_{B_C,B_\iota}(T')\right|^{-1},$$

$$T'(G;x,y) = -\left|M^{H,H'}_{B_C,B_C,0}(T')\right| \cdot \left|M_{B_C,B_C}(T')\right|^{-1}.$$

Determinant formulas of size $b_p + 1$ if $P_C = \{Q_1, \ldots, Q_p\}$.

Let $\overline{T}(G;q,v) = T\left(G; \frac{q}{v}+1, v+1\right) q^{c(E)} v^{|V|-c(E)} = \sum_{A \subseteq E} q^{c(A)} v^{|A|}$

contraction-deletion rule:

$\overline{T}(G;q,v) = q^{|V|}$ if $E = \emptyset$

$\overline{T}(G;q,v) = \overline{T}(G-e;q,v) + v \cdot \overline{T}(G/e;q,v)$ if $e \in E$

$\overline{T}(G;q,v)$ is not a T-G invariant on the class of graphical matroids

Determinant formula of size $b_n + 1$ for an n-cut (Negami)

Flow polynomial $F(G;k) = (-1)^{r^*(G)} T(G;0,1-k)$

$$F(G;k) = 1 \qquad \text{if } E = \emptyset$$

$$F(G;k) = 0F(G-e;k) \qquad \text{if } e \text{ is an isthmus of } G$$

$$F(G;k) = (k-1)F(G-e;k) \qquad \text{if } e \text{ is a loop of } G$$

$$F(G;k) = F(G/e;k) - F(G-e;k) \qquad \text{otherwise}$$

F is isthmus-smooth and $F_1^{\text{is}} = F$

$F(G;k) = 0$ if G has an isthmus.

Let $S_n = \Big\{ P = \{Q_1, \ldots, Q_i\} \in B_n; |Q_1|, \ldots, |Q_p| \geq 2 \Big\}$.

$\forall\, P \in S_n$: $[P, P]$ has no isthmus

Let $s_n = |S_n|$; $\quad s_1 = 0$, $s_0 = s_2 = s_3 = 1$, $\forall\, n \geq 4$,

$$s_n = 1 + \sum_{i=2}^{n-2} \binom{n-1}{i-1} s_{n-i}.$$

$M_{S_n,S_n}(F)$ is regular $\forall n \geq 2$

Theorem 4: Let C be an n-edge cut of a graph G. Then

$$\left| M^{H,H'}_{S_n,S_n}(F) \right| = 0, \text{ whence}$$

$$F(G;k) = -\left| M^{H,H'}_{S_n,S_n,0}(F) \right| \cdot \left| M_{S_n,S_n}(F) \right|^{-1}.$$

Determinant formula of size $s_n + 1$.

$M_{B,B}(F)$ is regular for each $B \subseteq S_n$.

Define $S_C = S_n \cap B_C$.

Theorem 5: Let C be an n-edge cut of a graph G. Then

$$\left| M^{H,H'}_{S_C,S_C}(F) \right| = 0, \text{ whence}$$

$$F(G;k) = -\left| M^{H,H'}_{S_C,S_C,0}(F) \right| \cdot \left| M_{S_C,S_C}(F) \right|^{-1}.$$

Determinant formula of size $s_p + 1$ if $P_C = \{Q_1, \ldots, Q_p\}$.

$$S_4 = \Big\{ \big\{\{1,2,3,4\}\big\},\ \big\{\{1,2\},\{3,4\}\big\},\ \big\{\{2,3\},\{1,4\}\big\},\ \big\{\{1,3\},\{2,4\}\big\} \Big\}$$

$$M_{S_4,S_4} = \begin{pmatrix} k^3-4k^2+6k-3 & (k-1)^2 & (k-1)^2 & (k-1)^2 \\ (k-1)^2 & (k-1)^2 & k-1 & k-1 \\ (k-1)^2 & k-1 & (k-1)^2 & k-1 \\ (k-1)^2 & k-1 & k-1 & (k-1)^2 \end{pmatrix}$$