Fourientations and the Tutte polynomial: some connections

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Overview

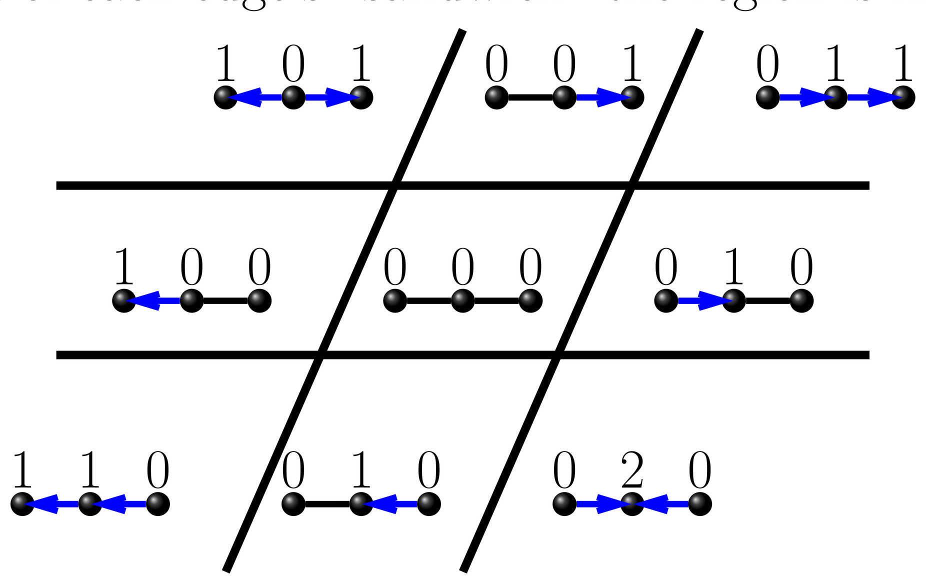
Fix $G := (V, E)$. **Spencer Backman will give a talk** on our main result: a Tutte polynomial formula enumerating **min-edge cut-cycle classes** of (generalized) orientations of G. Without defining the classes, here are two connections: **bigraphical arrangements**, and **monomizations of zonotopal algebras**.

Bigraphical arrangements

Let $A := (a_{e^\pm}) \in \mathbb{R}^{2|E|}_{>0}$ be a **parameter list**. The **bigraphical arrangement** associated to A is the collection of $2|E|$ hyperplanes:

$$\Sigma_G(A) := \{x_u - x_v = a_{e^\pm} \colon e^\pm = (u, v)\} \subseteq \mathbb{R}^{|V|}$$

There is a natural map $R \mapsto \mathcal{O}_R$ that sends a region of $\Sigma_G(A)$ to a **partial orientation** recording the third of each edge's "sandwich" the region is in:

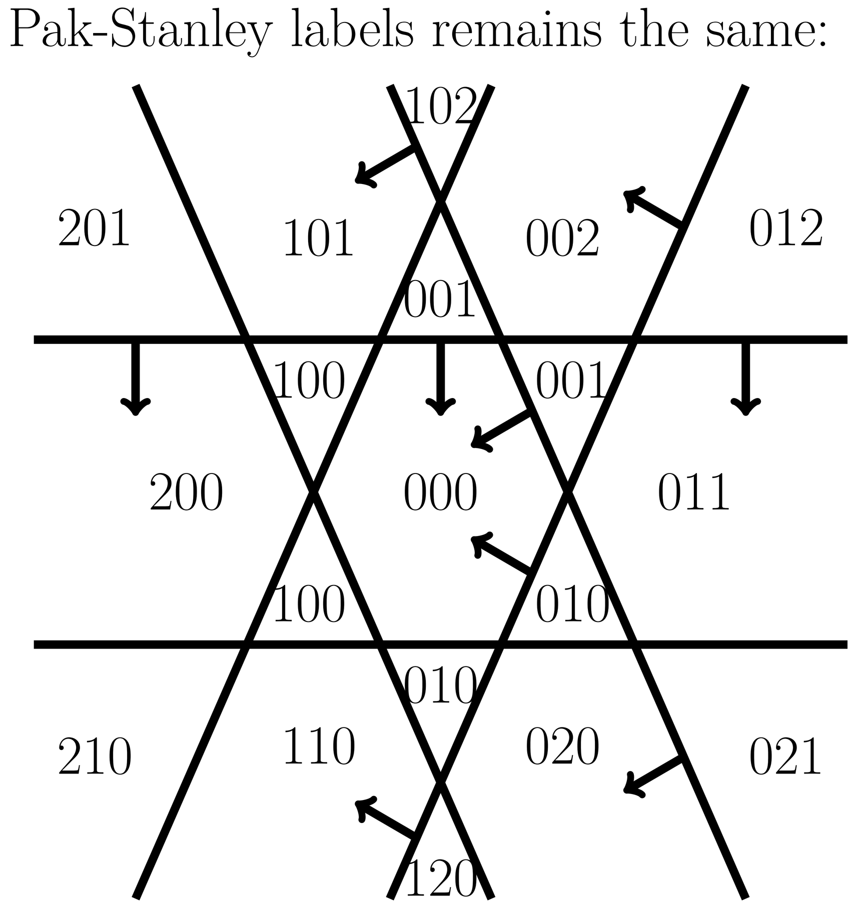


$R \mapsto \mathrm{indeg}(\mathcal{O}_R)$ is the **Pak-Stanley** labeling.

Theorem (H.-Perkinson)

For any A, the set of Pak-Stanley labels of $\Sigma_G(A)$ is the set of $G_\bullet$-parking functions.

As hyperplanes slide, regions come and go but the set of Pak-Stanley labels remains the same:



Exponential parameters

H.-Perkinson compute the numbers of (bounded) regions of $\Sigma_G(A)$ for a **generic parameter list** A:

$$r(\Sigma_G(A)) = 2^{|V|-1} T_G(3/2, 1);$$
$$b(\Sigma_G(A)) = 2^{|V|-1} T_G(1/2, 1).$$

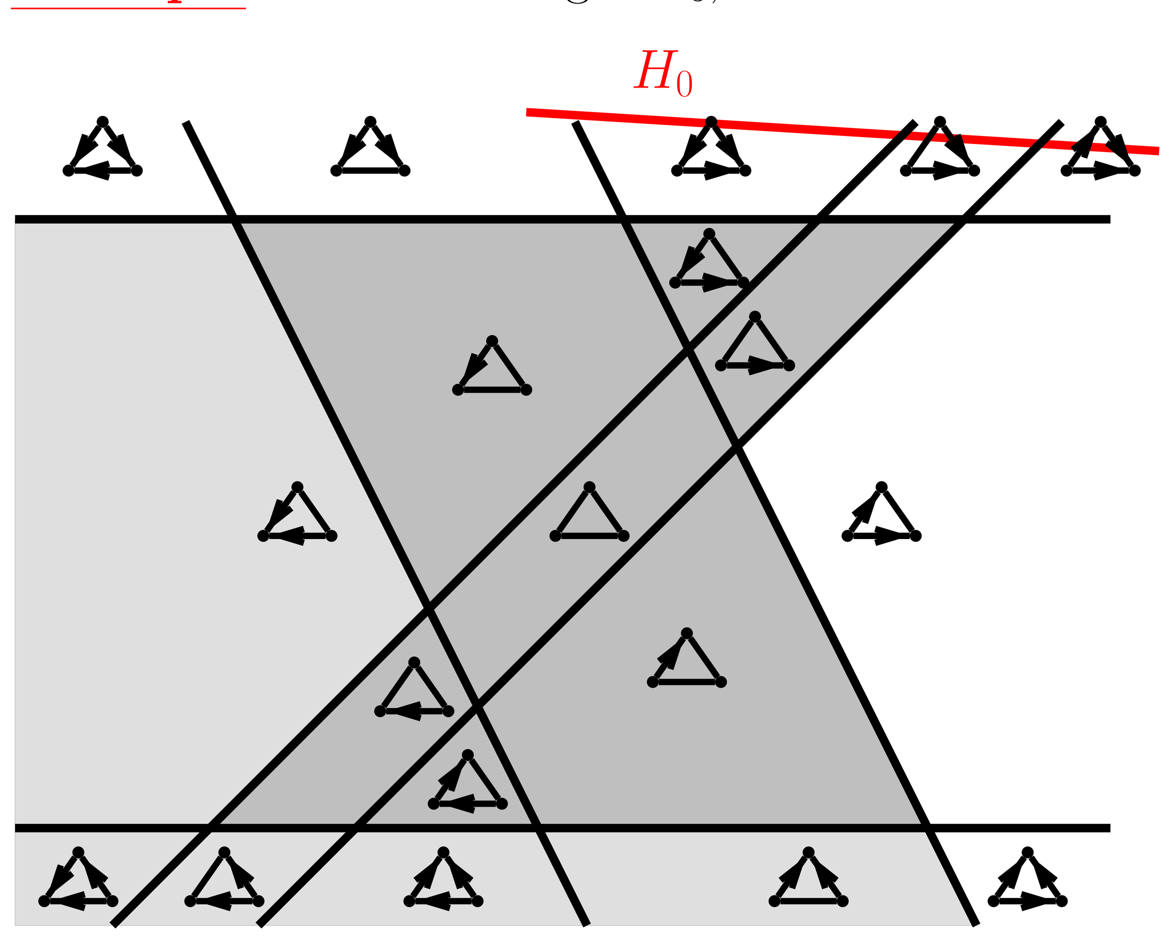
For edge order $e_1 < \cdots < e_m$, define the **exponential paramater list** $A^< := (a^<_{e^\pm})$ by $a^<_{e_i^\pm} := (1/2)^i$.

Theorem

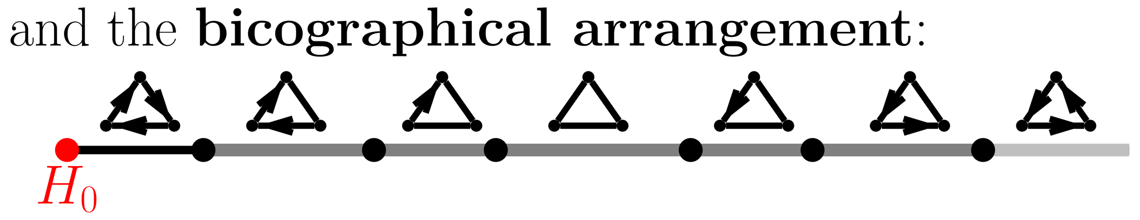
The map $R \mapsto \mathcal{O}_R$ is a bijection between:

- regions of $\Sigma_G(A^<)$ and **cycle neutral** po's;
- regions of $\Sigma_G(A^<)$ that avoid the hyperplane
  $$H_0 := \sum_{e^+=(u,v)} a^<_{e^+}(x_v - x_u) = -M$$
  and **cut minimal-cycle neutral** po's;
- bounded regions of $\Sigma_G(A^<)$ and **strongly connected-cycle neutral** po's.

Example: For the triangle K_3,



There is an analogous result for **cut neutral** po's and the **bicographical arrangement**:



G-parking functions

Now fix a **root** $q \in V$. Set $R := \mathbf{k}[x_u \colon u \in V \setminus \{q\}]$. The (G, q)**-parking function ideal** is

$$I_{(G,q)} := \langle \prod_{u \in U} x_u^{\#\{e=\{u,v\}, v \in U^c\}} \colon \emptyset \neq U \subseteq V \setminus \{q\} \rangle.$$

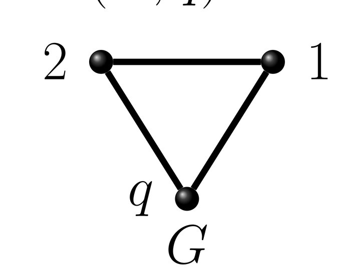
Denote the set of **acyclic,** q**-connected** partial orientations by $\mathcal{A}(G, q)$.

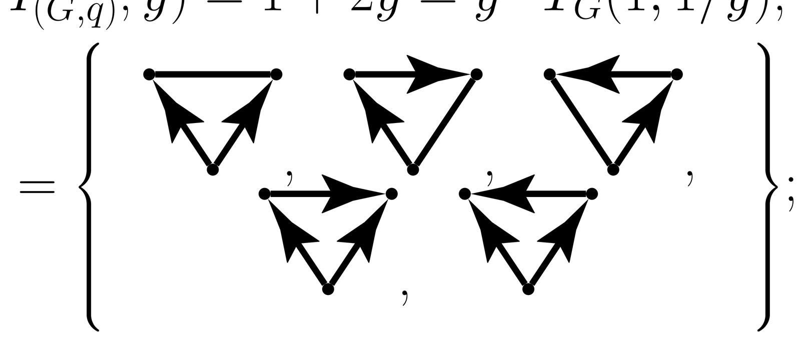
Thm. (Merino, Dhar, etc.)

For any (G, q),

- $\mathrm{Hilb}(R/I_{(G,q)}; y) = y^g \cdot T_G(1, 1/y)$;
- a linear basis of $R/I_{(G,q)}$ is
  $$\{\mathbf{x}^{\mathrm{indeg}(\mathcal{O})-1} \colon \mathcal{O} \in \mathcal{A}(G, q)\}.$$

Example: Suppose (G, q) is



- $I_{(G,q)} = \langle x_1^2, x_2^2, x_1x_2 \rangle$;
- $\mathrm{Hilb}(R/I_{(G,q)}; y) = 1 + 2y = y \cdot T_G(1, 1/y)$;
- $\mathcal{A}(G, q) =$  ;
- $\{1, x_1, x_2\}$ is indeed a basis of $R/I_{(G,q)}$.

Zonotopal algebras

For $r \geq -1$ define the **power form ideal**

$$J^r_{(G,q)} := \left\langle \left(\sum_{u \in U} x_u \right)^{\#E(U,U^c)+r} \colon \emptyset \neq U \subseteq V \setminus \{q\} \right\rangle.$$

The algebras $R/J^r_{(G,q)}$ for $r = +1, 0, -1$ are the **external**, **central**, and **internal** zonotopal algebras.

Thm. (Ardila-Postnikov, etc.)

- $\mathrm{Hilb}(R/J^{+1}_{(G,q)}; y) = y^g \cdot T_G(1 + y, 1/y)$;
- $\mathrm{Hilb}(R/J^{0}_{(G,q)}; y) = y^g \cdot T_G(1, 1/y)$;
- $\mathrm{Hilb}(R/J^{-1}_{(G,q)}; y) = y^g \cdot T_G(0, 1/y)$.

Monomizations

Monomization of an ideal: (nice) monomial ideal with same Hilbert series. For zonotopal algebras:

- Central: see left
- External: Desjardins constructs one, building off of Postnikov–Shapiro–Shapiro and P.-S.
- Internal: only known when G is **saturated**

Subparking functions

Fix a **q-rooted, ordered spanning tree** T. The (G, q, T)**-subparking function ideal** is

$$I^{-1}_{(G,q,T)} := \langle \prod_{u \in U} x_u^{\#\{e=\{u,v\}, v \in U^c, e \text{ not min. in } T \cap E(U,U^c)\}} \rangle.$$

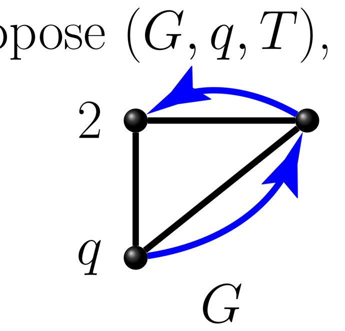
Denote the set of **acyclic,** q**-connected, cut neutral** partial orientations by $\mathcal{A}(G, q, T)$.

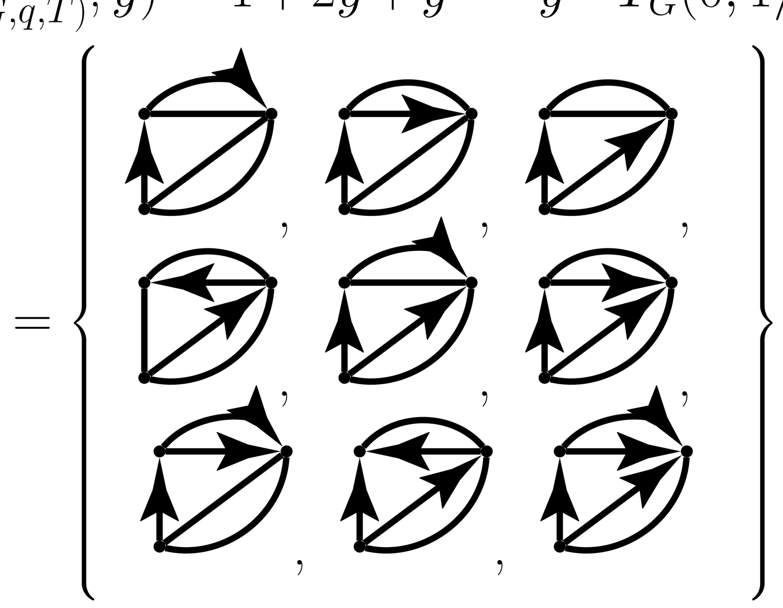
Conjecture

For any (G, q), there is T such that

- $\mathrm{Hilb}(R/I^{-1}_{(G,q,T)}; y) = y^g \cdot T_G(0, 1/y)$;
- a linear basis of $R/I^{-1}_{(G,q,T)}$ is
  $$\{\mathbf{x}^{\mathrm{indeg}(\mathcal{O})-1} \colon \mathcal{O} \in \mathcal{A}(G, q, T)\}.$$

Example: Suppose (G, q, T), with T blue, is



- $I^{-1}_{(G,q,T)} = \langle x_1x_2, x_2^2, x_1^3 \rangle$;
- $\mathrm{Hilb}(R/I^{-1}_{(G,q,T)}; y) = 1 + 2y + y^2 = y^3 \cdot T_G(0, 1/y)$;
- $\mathcal{A}(G, q, T) =$  ;
- $\{1, x_1, x_2, x_1^2\}$ is indeed a basis of $R/I^{-1}_{(G,q,T)}$.

Note: We have proved this conjecture when G is **outerplanar** (and maybe even **series-parallel**?)