Equivalence of Embedded Graphs and Twisted Duality in Hypermaps

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Introduction

A hypergraph is a graph in which an edge can be incident to more than two vertices.

We want to define what it means to embed a hypergraph in a surface. We do this by using a band decomposition

Definition. A *band decomposition* of a surface is a collection of $0-bands$, $1-bands$ and $2-bands$ which satisfy the following conditions:

1. Each band is adisk
2. Different bands only intersect on their boundaries.
3. The union of all the bands is the entire surface.
4. Each 1-band intersects with at least one 0-band.
5. Each 1-band intersects with at least 2-band.
6. Each class of band is pairwise disjoint i.e. no two 0-bands can intersect.

We can obtain a hypergraph from a band decomposition by taking the $0-bands$ to be the vertices and the $1-bands$ to be the edges. We say that this is an embedding of a hypergraph in a surface and call it a hypermap.

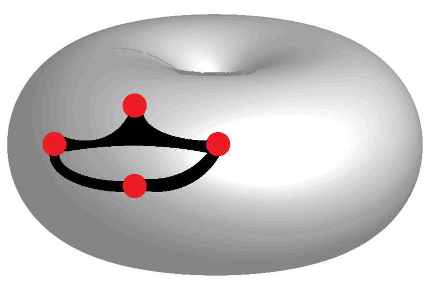

Figure 1: A band decomposition representation of a hypermap

Once we have this definition we can define another couple of representations of hypermaps which we will need for our proofs.

Definition. A *ribbon hypergraph* $H = (V(H), F(H))$ is a (possibly non-orientable) surface with boundary represented as the union of two sets of disks, a set $V(H)$ of $vertices$ and a set $E(H)$ of $edges$ such that

1. The vertices and edges intersect in disjoint line segments called *joints*
2. Each such joint lies on the boundary of precisely one vertex and precisely one edge

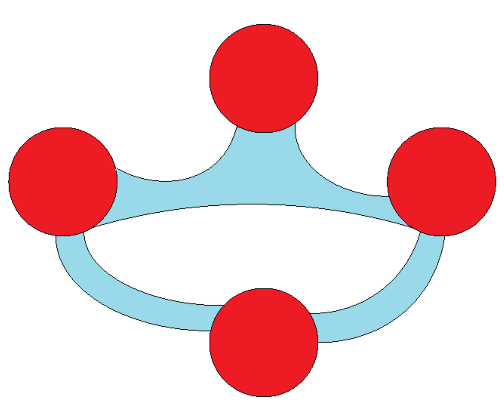

Figure 2: Ribbon Hypergraph

Definition. The *arrow presentation of a hypermap* consists of a set of closed curves, which represent the vertices, and sets of labelled arrows which represent the edges. The set of arrows are partitioned into cyclically ordered subsets such that each subset represents a single hyperedge, we will refer to each subset as an edge of the arrow presentation. For example given an edge e we label the subset of arrows that represent it $e_1, \ldots e_n$, where the subscript gives its position in the cyclic order.

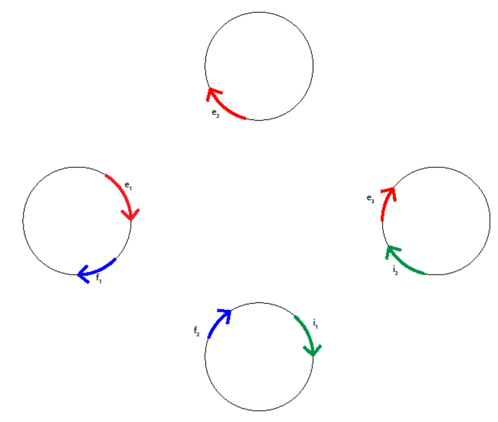

Figure 3: Arrow Presentation of a Hypermap

We also introduce the geometric dual and medial graph of a hypermap and Tait Hypergraphs which are defined as follows.

Definition. The *Geometric Dual* $H^* \subset \Sigma$ of a cellularly embedded hypermap $H \subset \Sigma$ is formed by relabbeling the $0-bands$ as $2-bands$ and vice versa in the band decomposition representation of the hypermap.

Definition. Given a cellularly embedded hypermap H we construct its *medial graph* H_m by for each edge e of H with cardinality k placing a vertex of degree $2k$ on the edge and then drawing the edges of the medial graph by following the face boundaries of H. Observe that H_m is a classical graph but unlike the existing medial graph does not have to be $4-regular$.

Definition. Let F be a checkerboard coloured cellularly embedded graph then

1. The *Blackface Hypermap* F_{hb} of F is the embedded hypergraph constructed by placing a new vertex in each black face, then for each of the vertecies of F, of degree $2k$, an edge of cardinality k is added to F_{hb} connecting the new vertices corresponding to the black faces which meet at this vertex.
2. The *Whiteface Hypermap* F_{hw} of F is constructed analogously by placing vertices in the white faces.

The two *Tait hypergraphs* are the hypermaps obtained by checkerboard colouring F and forming the whiteface and blackface hypergraphs.

We first consider equivalence as embedded hypergraphs and prove the following results.

Results

Theorem. *Let F be an even embedded graph. Then F is the medial graph of some embedded hypergraph H if and only if F is checkerboard colourable. If F is checkerboard colourable, then F is the embedded medial graph of precisely F_{hb} and F_{hw}.*

Theorem. *Let H be an embedded hypergraph and F be a checkerboard coloured embedded graph. Then*

1. $\{F_{hb}, F_{hw}\} = \{H \mid H_m = F\}$*, or, equivalently,* $H_m = F \iff H = F_{hb}$ *or* $H = F_{hw}$
2. $(F_{hb})_m = (F_{hw})_m = F$
3. $\{(H_m)_{hb}, (H_m)_{hw}\} = \{H, H^*\}$
4. $F_{hb} = (F_{hw})^*$
5. $\{H, H^*\} = \{G \mid G_m = H_m\}$*, or, equivalently,* $G_m = H_m \iff G \in \{H, H^*\}$

However we also want to consider equivalence of abstract graphs with the ultimate goal being to show the following:

Theorem. *Let G be an embedded hypergraph. Then*

$$Orb(G) = \{H \mid H_m \cong G_m\}$$

i.e.

$$H_m \cong G_m \iff H \in Orb(G)$$

Method of Proof

We introduce three operations that act on the arrow presentation of a hypermap.

Definition. *The dual with respect to an edge e* of H is $(H^{\delta(e)}) = H')$, where H' is formed from H as follows. Let e be an edge of cardinality k then in the arrow presentation of H there will be k labelled arrows $\{e_1, \cdots, e_k\}$, for $1 < i < k-1$ draw a line segment with an arrow from the head e_i to the tail of the e_{i+1} label this new arrow e_i, for $i = k$ draw an arrow from the head of e_k to the tail of e_1 and label this e_k and then delete the original arrows. The new arrows become arcs of new closed curves in the arrow presentation of H'. We call H' a partial dual of H

Definition. The *partial petrial* $H^{\tau(e^i)}$ is the hypermap obtained by reversing the direction of the arrow e^i in the arrow presentation of H.

Definition. The *partial permutation* $H^{\sigma(e^{(i,j)})}$ is the hypermap obtained by switching the cyclic order of arrows e^i and e^j in the arrow presentation of H.

Definition. We define two hypermaps G and H as *twisted duals* if one can be obtained from the other by using any combination of the local operations τ, σ and δ. We use *Orb(H)* to denote the set of twisted duals of a hypermap H.

We then introduce the cycle family graphs

Definition. An *arrow marked vertex state* is formed by replacing a vertex v of degree n with a set arrows connecting the half edges. We do this by deleting the vertex and part of the edges in a small neighbourhood around v we then connect the endpoits of the half edges with one another using arrows and give these arrows a cyclic labelling $v_1, \cdots, v_n$. The possible arrow marked vertex states for a given vertex include all permutations of connected pairs of half edges.

Definition. Let F be an even cellularly embedded graph. A *cycle family graph* of F is an embedded hypergraph obtained as the arrow presentation given by replacing each vertex with one of the possible arrow marked vertex states We let $\mathfrak{C}(F)$ denote the set of Cycle Family Hypergraphs of F.

and show that

Theorem. *Let F be an even abstract graph and let $\tilde{F}$ be any embedding of F. Then*

$$\mathfrak{C}(\tilde{F}) = \{G \mid G_m \cong F\}$$

i.e.

$$G_m \cong F \iff G \in \mathfrak{C}(\tilde{F})$$

for some embedding $\tilde{F}$ of F.

and

Theorem. *Let G be an embedded hypergraph. Then the cycle family graphs of its hyper medial graph (G_m) are exactly its twisted duals, i.e.*

$$\mathfrak{C}(G_m) = Orb(G).$$

and by putting these two theorems together we get our result.

References

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