New results for the Merino-Welsh conjecture

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Introduction

Let $T_G(x,y)$ be the Tutte polynomial of a graph G. Merino and Welsh conjectured that for a 2-connected loopless multigraph G,

$$T_G(1,1) \le \max\{T_G(2,0), T_G(0,2)\}.$$

The three numbers $T_G(1,1), T_G(2,0), T_G(0,2)$ are precisely the numbers of spanning trees, acyclic orientations and totally cyclic orientations respectively, and we write them as $t(G), a(G)$ and $c(G)$ for short. Thus, the conjecture claims $t(G) \le \max\{a(G), c(G)\}$.
Two variants of the Merino-Welsh conjecture have been studied, which are the following:

- $t(G)^2 \le a(G)c(G)$ (multiplicative version)
- $T_G(x,y)$ convex on the line segment between (2,0) and (0,2). (convex version)

I present new results for each of these three conjectures.

The original version

Thomassen proved the following:
For G with n vertices and m edges,

- $t(G) \le \left(\frac{2m}{n}\right)^n$,
- $a(G) \ge 2^{n-1}$,
- $c(G) \ge 2^{m-n+1}$.

It follows that

- $t(G) \le a(G)$ if $m \le 1.066n$,
- $t(G) \le c(G)$ if $m \ge 4(n-1)$.

I proved

1. $t(G) \le a(G)$ if $m \le 1.29(n-1)$,
2. $t(G) \le c(G)$ if G is 3-edge-connected and $m \ge 3.58(n-1)$.

The proof uses the fact that these orientations have many **flippable edges**.

The convex version

I proved that the convex version holds for minimally 2-edge-connected graphs.

Proof:
Let M be a matroid in which each element is in a circuit of size 2.
Then the Tutte polynomial $T_M(x,y)$ is

$$\sum_{a,b\ge 0} C_{a,b}\, y^a (x+y)^b$$

where $C_{a,b} \in \mathbb{Z}_{\ge 0}$.
The proof is by induction on the number of **parallel classes**.
Hence, $T_M(2-y,y) = \sum_{i\ge 0} C'_i y^i$, $C'_i \in \mathbb{Z}_{\ge 0}$, which is convex in $0 \le y \le 2$.
Since $T_{M^*}(x,y) = T_M(y,x)$, the dual matroids also have convex Tutte polynomials, and these include all minimally 2-edge-connected graphs.

Further topics #1

Thomassen asked the following:
For G loopless, 2-connected, $n = |V|$, $m = |E|$,

- Is $a(G) \ge t(G)$ for $m \le 2n-2$?
- Is $c(G) \ge t(G)$ for $m \ge 2n-2$?

Noble and Royle found counterexamples with $m = 2n-2$ for both. Actually, their example can be modified to $2n - \log_2 n$ edges.

Thus, we may ask instead for an arbitrarily small constant $\epsilon > 0$ such that

- $a(G) \ge t(G)$ for $m \le (2-\epsilon)n$,
- $c(G) \ge t(G)$ for $m \ge (2+\epsilon)n$

for sufficiently large n.

The computational complexities of related problems

Assuming the Merino-Welsh conjecture, the following problem is answered in constant time.

- **Instance :** 2-connected multigraph G, **Query :** $t(G) \le \max\{a(G), c(G)\}$.

But without the 2-connected condition, the following problem is #P-complete.

- **Instance :** multigraph G, **Query :** $t(G) \le \max\{a(G), c(G)\}$.

The reason is because it is equivalent to the following set of two problems

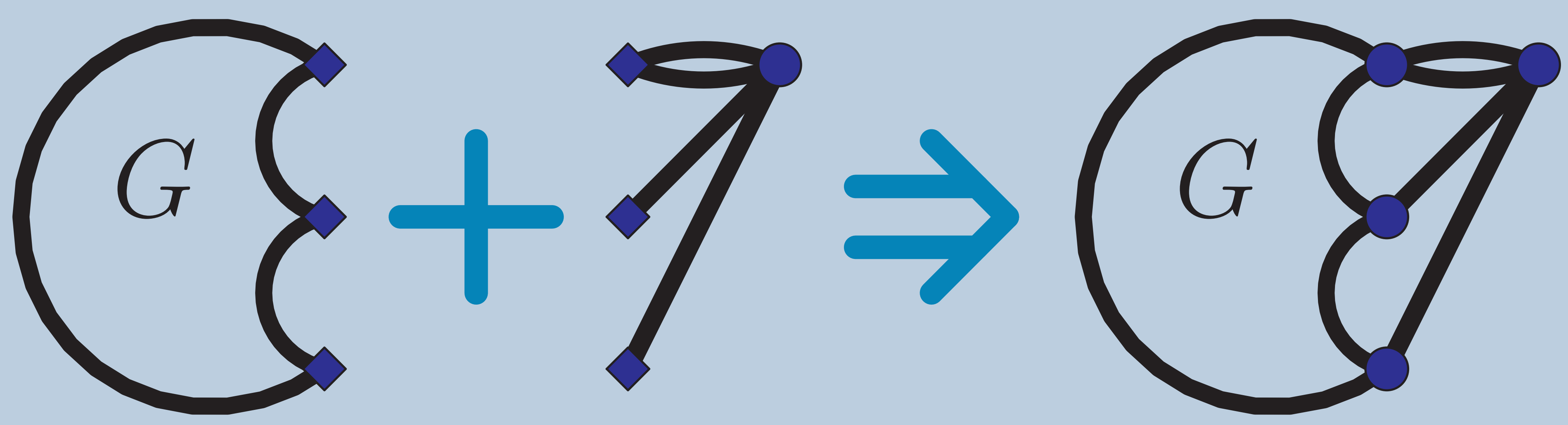
- **Instance :** multigraph G, **Query (1) :** $t(G) \le a(G)$, **Query (2):** $t(G) \le c(G)$.

which are respectively polynomial-time equivalent to calculating $a(G)$ and $c(G)$ precisely.

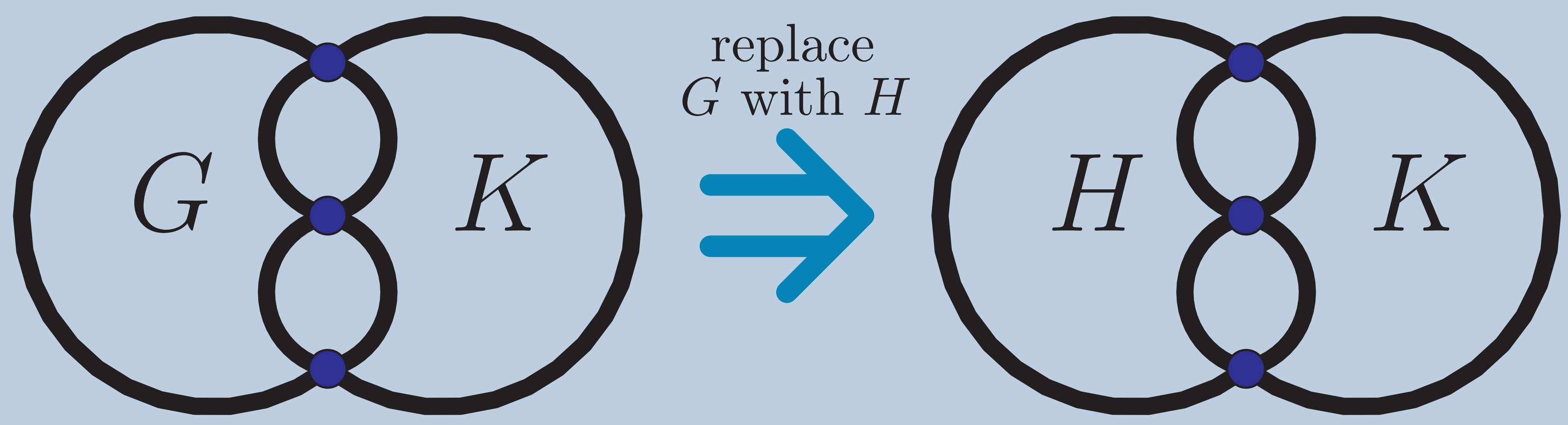
The multiplicative version

I designed an algorithm for graphs of bounded treewidth or pathwidth, and using it,
I verified the multiplicative Merino-Welsh conjecture for graphs of pathwidth at most 3.

Proof method:
Bounding the treewidth or pathwidth provides a natural construction of the graphs, by adding vertices one by one.



I found a sufficient condition between two graphs G and H that, for any K,
the multiplicative conjecture holds for $G \oplus K$ if it holds for $H \oplus K$.



That is, G is **replaceable** by H. It extends Noble and Royle's idea for series-parallel graphs.
With a computer search, I have 5242 graphs that replace all 2-connected graphs of pathwidth at most 3. And these graphs confirm the multiplicative conjecture for pathwidth ≤ 3.

The algorithm is applicable to treewidth (or pathwidth) $\le k$ for any k, but I do not know the finiteness of the minimal list for other k.

Further topics #2

• We may ask whether the above algorithm for multiplicative version yields a finite list for other bounds of treewidth. The finiteness seems to be related with the concept of **protrusion** in parametrized complexity theory.

• Also, extending Thomassen's idea, I believe that

$$\frac{a(G)}{t(G)} \le \frac{a(G-e)}{t(G-e)}$$

for every loopless connected graph G and non-bridge e. That is, as the graph gets sparser, the acyclic orientations gets more dominant over the spanning trees. Details in my thesis.