Abstracts of Talks

Graph Fourientations and the Tutte Polynomial

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Graph fourientations generalize graph orientations by the inclusion of unoriented and bidirected edges. We present an intersection lattice of 64 fourientation classes determined by cut-cycle properties, which admit trivariate generating functions of the form

$$(k+m)^{n-1}(k+l)^g T\left(\frac{\alpha k+\beta l+m}{k+m}, \frac{\gamma k+l+\delta m}{k+l}\right)$$

with $\alpha, \gamma \in \{0,1,2\}$ and $\beta, \delta \in \{0,1\}$. These enumerations are proven by a single uniform deletion-contraction argument, and they specialize to a host of well-known results relating graph orientations and the Tutte polynomial. We will briefly mention connections to bigraphical arrangements, Lawrence ideals, Riemann-Roch theory for graphs, and zonotopal algebras. This is joint work with Sam Hopkins.

References.

[1]S. Backman, Riemann-Roch Theory for Graph Orientations, *arXiv:1401.3309* January 2014
[2]S. Backman, Partial Graph Orientations and the Tutte Polynomial, *arXiv:1408.3962* August 2014
[3]S. Backman and S. Hopkins, Fourientations and the Tutte Polynomial, *arXiv:1503.05885* March 2015
[4]S. Hopkins and D. Perkinson, Bigraphical Arrangements *arXiv:1212.4398*. Forthcoming *Trans. Amer. Math. Soc.*, December 2012.

Cyclic flats of matroids and their connections to Tutte polynomials and other matroid invariants

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A *flat* in a matroid M is a subset X of $E(M)$ for which $r(X \cup y) > r(X)$ for all $y \in E(M) - X$, where r is the rank function of M. To specify M, it suffices to give its flats. The flats of M, ordered by inclusion, form a geometric (i.e., atomic and semimodular) lattice.

A flat is *cyclic* if it is a (possibly empty) union of circuits. The cyclic flats alone do not specify M, but the set $E(M)$ and the pairs $(X, r(X))$, as X ranges over the cyclic flats of M, give a relatively compact way to specify M. The cyclic flats of a matroid also form a lattice under inclusion, and, up to isomorphism, all finite lattices can be realized this way.

Jens Eberhardt recently showed that the Tutte polynomial of a matroid is determined by the abstract lattice of its cyclic flats along with the rank and size of the set corresponding to each lattice element (without giving the set). Many non-isomorphic matroids can share this data. We will sketch an alternative proof of Eberhardt's result and an extension of it to Derksen's $\mathcal{G}$-invariant. (This is joint work with Joseph Kung.)

If time permits, we will also touch on other areas of matroid theory in which viewing matroids in terms of cyclic flats has proven useful.

Delta-matroid polynomials and the symmetric Tutte polynomial

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Delta-matroids form a natural generalization of matroids and were introduced by Bouchet in 1987. In an effort to find a common generalization of (1) the Tutte polynomial for matroids restricted to the diagonal (i.e., the case $x = y$) and (2) the interlace polynomial for graphs, we obtain a polynomial for delta-matroids. Many properties of the Tutte polynomial and the interlace polynomial hold for this general delta-matroid polynomial. Other polynomials, such as the Penrose polynomial, can also be generalized in this way. Joint work with Hendrik Jan Hoogeboom.

Tutte polynomial and orbit-counting

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abstract Specialisations of the Tutte polynomial solve a number of counting problems: bases, spanning sets, independent sets, graph colourings and acyclic orientations, etc.

The cycle index of a permutation group enables us, by specialisation, to count orbits of the group on functions on its domain.

It is natural to try to combine these two ideas, to count orbits of the automorphism group of a matroid on functions satisfying some matroid-like structural restriction, such as colourings and acyclic orientations of graphs.

I will outline a couple of approaches which have been partially successful.

Krushkal polynomial of graphs on surfaces

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The Krushkal polynomial is the most general polynomial invariant of graphs on surfaces. It was introduced in [1]. A collection of edges forms a spanning *quasi-tree* if the regular neighborhood of the corresponding spanning subgraph has one boundary component. In the case of plane graphs it is just a spanning tree. The set of edges of a spanning quasi-trees is a feasible set of the corresponding delta-matroid. If the edges of the graph are ordered then one can introduces activities relative to a spanning quasi-tree [2]. The goal of my talk is to present a quasi-tree expansion of the Krushkal polynomial discovered by Clark Butler in [3].

References.

[1] V. Krushkal, , Graphs, links, and duality on surfaces., *Combin. Probab. Comput.*, 20, 267–287 (2011).

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A splitter theorem for internally 4-connected graphs

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Two powerful inductive tools for dealing with 3-connected graphs are Tutte's Wheels Theorem and Seymour's Splitter Theorem. Tutte's Wheels Theorem asserts that, for a 3-connected graph G, there is an edge e in G such that the deletion or contraction of e from G is 3-connected and simple unless G is a wheel. By Seymour's Splitter Theorem, a 3-connected graph G with a 3-connected proper minor H has an edge e such that the deletion or contraction of e from G is 3-connected and simple with a minor isomorphic to H, unless H is a wheel. In this talk we present the analogues of these two theorems adapted to the class of internally 4-connected graphs. This is a corollary of a more general result that we have proved for binary matroids.

A general notion of activity for the Tutte polynomial

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In the literature can be found several descriptions of the Tutte polynomial of graphs, especially in terms of activities. The first notion of activity is due to Tutte himself, and it requires to linearly order the edges [1]. Several decades later, Bernardi described his via an embedding of the graph into a surface [2]. In this talk, we show that other notions of activity can be imagined and they can all be embodied in a same notion, which we call Δ-*activity.* We also see that the main properties satisfied by the classical notions of activity are also shared by the Δ-activity.

References.

[1] W.T. Tutte, *A Contribution on the Theory of Chromatic Polynomial*, Can. Journal of Math. 6., 80–91 (1954).

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A Tutte polynomial for edge- and vertex- weighted graphs, list coloring, and the zero-temperature Potts model

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One of the many fruitful connections between combinatorics and physics arises from the recognition that, with constant interaction energy and in the absence of an external field, the classical Tutte polynomial of graph theory and the Potts model of statistical mechanics are the same object. Furthermore, the chromatic polynomial corresponds to the zero-temperature limit of the antiferromagnetic Potts model.

We describe a generalization of the Tutte polynomial to edge- and vertex-weighted graphs that lifts the classical relation to the much broader setting of Potts models with external fields. This leads to a new and unexpected connection between list coloring in graph theory and the zero-temperature Potts model with external fields. This connection not only provides new opportunities to further reap the rewards of cross-pollination between combinatorics and physics, but it also provides a formalization for, and gives an established graph theoretical foundation to, work in this area from a physics perspective.

This is joint work with Iain Moffatt, Royal Holloway University of London.

Minors and Tutte invariants for alternating dimaps

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The foundations of the modern theory of minors, in graphs and matroids, were laid by Tutte, and have their historical roots in his famous 1940 paper (with Brooks, Smith and Stone) on "squaring the square".

Tutte published a lesser known paper, on "triangulating the triangle", in 1948, which introduced a generalisation of duality, called *triality* or *trinity*, which applies to *alternating dimaps* (i.e., orientably embedded digraphs in which the edges incident at a vertex are directed alternately into, and out of, the vertex).

This talk will describe a theory of minors for alternating dimaps, based on three fundamental operations related by Tutte's triality. The theory has many analogies with standard minor theory. However, these minor operations are non-commutative. This makes their theory more difficult, but we are still able to establish some of the kinds of results one would hope for in a theory of minors, such as excluded minor characterisations and enumerative invariants that obey linear recurrence relations analogous to the deletion-contraction relation for the Tutte polynomial.

References.

[1] G. E. Farr, Minors for alternating dimaps, preprint, 2013,
http://arxiv.org/abs/1311.2783.

On Zero-free Intervals of Flow Polynomials

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Let G be a bridgeless graph and $W(G)$ be the set of vertices in G of degrees larger than 3. For any integer $k \geq 0$, let ξ_k be the supremum in $(1, 2]$ such that $F(G, \lambda)$ and $(-1)^{|E(G)|-|V(G)|+b(G)-1}$ have the same sign for all real $\lambda \in (1, \xi_k)$ and for all connected graphs G with $|W(G)| \leq k$, where $F(G, \lambda)$ is the flow polynomial of G and $b(G)$ is the number of blocks in G. We show that ξ_k can be determined by considering a finite set of graphs G with $|W(G)| = k$, and hence we find that $\xi_k = 2$ for $= 0, 1, 2$, $\xi_3 = 1.430\cdots$, $\xi_4 = 1.361\cdots$ and $\xi_5 = 1.317\cdots$, where the last three numbers are the real zeros of $\lambda^3 - 5\lambda^2 + 10\lambda - 7$, $\lambda^3 - 4\lambda^2 + 8\lambda - 6$ and $\lambda^3 - 6\lambda^2 + 13\lambda - 9$ in the interval $(1, 2)$ respectively.

We also show that if $W(G)$ is dominated by one component of $G - W(G)$, then $F(G, \lambda)$ has no zeros in $(1, 2)$.

References.

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Constructing the Tutte polynomial via polyhedral or algebraic geometry

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I will explain, from two points of view, a new construction for the Tutte polynomial of a matroid. The former point of view extracts it from counting lattice points in the matroid polytope and related polyhedra; the latter, from K-theory classes of subvarieties of a Grassmannian. In neither sort of geometry will expertise be assumed. The construction reveals connections to other invariants, such as an invariant of Speyer notionally counting the faces in a decomposition into series-parallel matroids. This work is joint with David Speyer.

A survey on the active bijection in graphs, hyperplane arrangements and oriented matroids.

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The active bijection maps any directed graph, resp. signed hyperplane arrangement or oriented matroid, on a linearly ordered edge set, resp. ground set, onto one of its spanning trees, resp. bases. It relates all spanning trees to all orientations of a graph, all bases to all reorientations of an hyperplane arrangement or, more generally, an oriented matroid. It preserves activities: for bases in the sense of Tutte, for orientations in the sense of Las Vergnas, yielding a bijective interpretation of the equality of two expressions of the Tutte polynomial. It can be mathematically defined in a short way, and can be built, characterized, particularized, or refined in several ways.

For instance, we get a bijection between bounded regions (bipolar orientations in the case of a graph) and bases with internal activity one and external activity zero, which can be seen as an elaboration on linear programming. We get a decomposition of into bounded regions of minors of the primal and the dual, which also has a counterpart for decomposing matroid bases, and yields an expression of the Tutte polynomial using only beta invariants of minors. We also get an activity preserving bijection between no-broken-circuit subsets and regions, a classical bijection between increasing trees and permutations (in the case of the complete graph), a refinement in terms of four variable expansions of the Tutte polynomial...

It is the subject of a series of paper joint with Michel Las Vergnas.

Graph polynomials by counting graph homomorphisms

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The chromatic polynomial evaluated at a positive integer n is equal to the number of homomorphisms to K_n, the complete graph on n vertices. Many important graph polynomials are likewise determined by counting homomorphisms to a sequence of (multi)graphs, such as the Tutte polynomial and the independence polynomial.

In [3] a powerful construction is given that produces graph polynomials in this way by generating suitable sequences of simple graphs (like (K_n) for the chromatic polynomial). To illustrate the method, some potentially interesting generalizations of the chromatic polynomial will be constructed, one related to fractional colourings and the other to circular colourings.

References.

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[2] D. Garijo, A.J. Goodall and J. Nešetřil, Polynomial graph invariants from homomorphism numbers, 40pp. Preprint: arXiv: 1308.3999 [math.CO]. Updated version (2015) submitted.

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Design Choices in Algorithm Development to Compute Tutte Polynomial

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The progress in computing Tutte polynomials has made it possible to deal with increasingly large examples. To understand how these computations are carried out, the talk will carefully examine the design decisions of the code of the Haggard-Pearce-Royle algorithm to describe the choice points and indicate other possible choices. It is the hope of this talk that others will see how results involving Tutte polynomials can lead to better choices in the algorithm or indicate why the current choices are the "right" ones. There is no reason that the choices should not change dynamically to reflect the changes in the structure of the graph during the computation. At his point there is no clear indication of how an algorithm might be structured to carry out different options to reflect recognizable changes in the structure of the graph during the computation.

Further work on using a grid of computers to compute the Tutte polynomials for 9-cages will be mentioned.

Graph Polynomials Motivated by Gene Assembly

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The interlace polynomial was discovered by Arratia, Bollobás, and Sorkin by studying DNA sequencing methods. Its definition can be traced from 4-regular graphs (the Martin polynomial), to circle graphs and finally to arbitrary graphs.

Our interest in these polynomials came from the study of ciliates, an ancient group of unicellular organisms. They have the remarkable property that their DNA is stored in two vastly different types of nuclei. The two versions of the versions of the gene can be elegantly modelled using a 4-regular graph.

We give an overview of the polynomials involved, their basic properties, and their relation to the Tutte polynomial.

Joint work with Robert Brijder.

References.

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[2] R. Brijder, H.J. Hoogeboom, The Algebra of Gene Assembly in Ciliates. In: *Discrete and Topological Models in Molecular Biology* (N. Jonoska, M. Saito, eds.), Natural Computing Series, Springer, 289–307, 2014.

Euler Characteristics and Chromatic Polynomials

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We review an old construction in which, given a graph, one constructs a family of manifolds whose Euler characteristics are the values of the chromatic polynomial of the graph at various integers. The manifolds are simple generalisations of configuration spaces. Then we consider recent attempts to extend this to the Tutte polynomial.

Critical manifolds and exact solvability from a topologically weighted Tutte polynomial

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The critical temperature is only known analytically for the simplest two-dimensional models (Ising model), or for more complicated models (Potts and $O(n)$ vector models) on the simplest possible lattices. The known critical temperatures are invariably given by simple algebraic curves. These results can be derived by non-rigorous means, such as duality arguments or integrability results, and some have recently been proved by SLE related techniques.

For the Potts and (bond or site) percolation models on any desired two-dimensional regular lattice we define a graph polynomial (a variant of the Tutte polynomial in the Potts case) that distinguishes between various topological sectors of the transfer matrix. Its roots turn out to give very accurate approximations to the critical temperatures, or even yield the exact result in the exactly solvable cases. This polynomial depends on a basis (unit cell) and its embedding into the infinite lattice. As the size of the basis is increased the approximation becomes increasingly accurate. This, on the one hand, gives strong evidence that the critical temperature for the lattices with no known analytical solution may not be algebraic numbers, and that conformal invariance will not have any counterpart in finite size (discrete holomorphicity). On the other hand, the method determines the critical temperature to unprecedented accuracy. For instance, we find $p_c = 0.52440499916737(4)$ for bond percolation on the kagome lattice, and $p_c = 0.5927460508\,03(2)$ for site percolation on the square lattice. It also shows that the phase diagram of the Potts model in the antiferromagnetic regime has an intricate and highly lattice-dependent structure.

Our method can be extended to O(n) models of loops, again providing exact results for solvable models and highly accurate approximations for models which are not. For self-avoding walks on the square lattice we thus find the critical monomer fugacity $z_c = 0.3790522777533(2)$, which is more accurate than the best estimates coming from series analysis.

Finally, the method can be applied to quenched random systems. In models having a gauge symmetry, such as the $\pm J$ random-bond Ising model and the q-state Potts gauge glass, it provides very accurate estimates of the so-called Nishimori point. For instance, in the Ising case the critical probability for a frustrated bond is found to be $p_N = 0.10929(2)$.

The computational complexity of approximating the Tutte polynomial

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The computational complexity of exact evaluation of the Tutte polynomial of a graph is well understood, thanks to the seminal work of Jaeger, Vertigan and Welsh [3], and succeeding authors. For each $(x, y) \in \mathbb{C}^2$, the function $G \mapsto T(G; x, y)$ is either polynomial-time computable or is #P-complete. We are very far from having such precise information about the complexity of approximating the function $G \mapsto T(G; x, y)$ within specified relative error.

We know from a randomised bisection technique due to Valiant and Vazirani that every problem in #P can be approximated with small relative error in polynomial time using an NP-oracle. One might therefore expect a dichotomy, similar to that that described by Jaeger et al., to apply to approximate computation: that for each $(x, y) \in \mathbb{C}^2$, the function $G \mapsto T(G; x, y)$ is either polynomial-time approximable or is essentially NP-complete to approximate.

In fact the situation is seemingly much more complicated. Some points (x, y) of the Tutte plane are #P-hard (i.e., "very hard") to approximate [2]. (The Valiant-Vazirai method does not apply, since the Tutte polynomial is a weighed counting problem, and not strictly a member of #P.) Other points are thought to be hard to approximate but are not known to be NP-hard; instead they are hard for some potentially smaller complexity class, which contains #BIS (counting independent sets in a bipartite graph) as a complete problem [1]. Yet other points are completely unresolved.

This is joint work with Leslie Goldberg, Oxford.

References.

[1] Leslie Ann Goldberg and Mark Jerrum. Approximating the partition function of the ferromagnetic Potts model. *J. ACM*, 59(5):Art. 25, 2012.

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From Broken Circuits to Monte Carlo: Approximating the Chromatic Polynomial and Its Friends

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It is known that the Tutte polynomial $T_M(x, y)$ is difficult to approximate, much less compute precisely, for most (x, y). We explore a Monte Carlo-type algorithm for approximating the chromatic polynomial, and examine questions of variance that arise in this context. We further show how this method can be extended to higher-dimensional equivalents of the chromatic polynomial and other Tutte polynomial specializations.

Splitting Formulas for Tutte-Grothendiek Invariants on Graphs

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Tutte-Grothendieck (shortly T-G) invariants are mappings from a class of matroids to a commutative rings that can be recursively characterized by contraction-deletion rules. Well known examples are Tutte, chromatic, tension and flow polynomials.

We transform T-G invariants on matroids so that the contraction-deletion rule for isthmuses or loops coincides with the general case and discuss their duality. Using these transformations, we introduce splitting formulas for T-G invariants on graphs as follows. Suppose that an edge cut C divides a graph G into two parts G_1, G_2 and $\mathcal{G}_1$, $\mathcal{G}_2$ are the sets of minors of G whose edge sets consist from C and edges of G_1, G_2, respectively. We study determinant formulas evaluating a T-G invariant of G from the T-G invariants of graphs from $\mathcal{G}_1$ and $\mathcal{G}_2$. In particular we deal with Tutte and flow polynomials on general and planar graphs.

Graph polynomials and matrix integrals

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Feynman graphs have been introduced to provide power series expansions of perturbations of Gaußian integrals in quantum field theory. They are graphs with the additional data of a bilinear form on the edges and higher rank tensors on the vertices. The contractions of the tensors with the bilinear forms yield the weight of the graph. We will present some examples, based on expansions of matrix integrals, whose weights are related to the transition and Bollobás-Riordan polynomials.

Syzygies between Tutte polynomials of freedom matroids

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The theory of G-invariants of matroids implies that every Tutte polynomial can be expressed as a linear combination of Tutte polynomials of freedom or nested matroids. This expression is not unique. We find a generating set of linear relations between Tutte polynomials of freedom matroids. This generating set is parametrized by intervals of height 2 in the weak order on freedom matroids. From the relations, we obtain two bases of Tutte polynomials of freedom matroids for all Tutte polynomials.

Splines, Integer Points in Polytopes, and the Tutte polynomial

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Box splines are compactly supported, multivariate piecewise polynomial functions that have been used and studied in approximation theory since the 1980s. Many of their properties are captured by the underlying matroid. In particular, the specialisations $T(1, q)$ and $T(0, q)$ of the Tutte polynomial play an important role in the theory.

Vector partition functions, which count the number of integer points in certain parametric polytopes, can be thought of as discrete analogues of box splines. Some of their properties are captured by the Tutte polynomial of the underlying arithmetic matroid.

In this talk I will introduce all the objects mentioned above and talk about some of my recent results, highlighting connections with the Tutte polynomial.

Semantic vs Syntactic Properties of Graph Polynomials

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Graph polynomials are graph invariants with values in a polynomial ring. Their presentation depends on the choice of a basis of basic polynomials and the choice of its coefficients. The situation is similar to the situation in linear algebra. Matrices represent linear mappings, but their presentation depends on the choice of a basis in the underlying vector spaces. Properties of matrices are properties of the linear maps, if they are invariant under matrix similarity. Otherwise they are merely properties of the particular matrix presentation, in other words they are syntactic properties. As an example, $\det(M) = 0$ is a semantic property of a matrix M, whereas being triangular is not. Two graph polynomials P, Q are semantically equivalent (aka equivalent by their distinguishing power or d.p.-equivalent) relative to a graph property Φ if whenever given any two graphs $G, H \in \Phi$ we have that $P(G) = P(H)$ iff $Q(G) = Q(H)$. Properties of graph polynomials are properties of the underlying graphs (semantic properties) if they are invariant under d.p.-equivalence of graph polynomials. Otherwise they are syntactic properties. As an example, the characteristic polynomial $P(G; \lambda)$ of G determines the number of triangles in G is a semantic property of $P(G; \lambda)$, whereas the fact that the absolute value of the coefficient of $\lambda^{|V(G)|-3}$ of P is twice the number of triangles of G, is a syntactic property.

In this talk we discuss which properties of graph polynomials, frequently studied in the literature and definable in some natural formalism (second order logic), are invariant under d.p.-equivalence, and which are not.
(This talk is based on published and unpublished work with E.V. Ravve)

References.

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My personal journey through the chip firing game

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In this talk I will present three results that gravitate around the chip firing game, some closer than others. Firstly, I will introduce a solitaire game, then I will describe the main structure that I'm interested in, the critical configurations of the game. The enumeration of these configurations, somewhat surprisingly, is related to the Tutte polynomial. But the connection with the matroid associated to the game runs more deep. I will argue this by presenting a result about matroidal complexes. Finally, I will return to the game and its configurations via Algebraic Geometry.

Hopf algebras and topological Tutte polynomials

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There are three versions of the Tutte polynomial for graphs embedded in surfaces: M. Las Vergnas' 1978 polynomial, B. Bollobás and O. Riordan's 2002 ribbon graph polynomial, and V. Kruskal's polynomial from 2011. Why are there three different "Tutte polynomials" of graphs in surfaces? To answer this question, I will propose a general Hopf algebraic framework for Tutte-like graph polynomials. I will go on to describe how a Hopf algebra, and therefore a "Tutte polynomial", can be obtained in a canonical way from sets of combinatorial objects equipped with "minors", showing that numerous graph polynomials in the literature, including the three mentioned above, arise from this canonical construction and that some of their properties can be deduced from their Hopf algebra structures. The aim of this work is to provide an approach for studying graph polynomials en masse, rather than individually.

This is joint work with Thomas Krajewski and Adrian Tanasa.

Theta Graphs, Bicliques and the Alpha Conjectures

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Understanding which numbers can be roots of chromatic polynomials is not only of theoretical interest but has applications in statistical mechanics where the partition function generalises the chromatic polynomial. Phase transitions can occur only at a real limit point of the complex roots of this function.

Which algebraic numbers can be roots of the chromatic polynomial? While this question largely remains open, we present results on roots of chromatic polynomials of families of graphs and look at open conjectures.

Families of bicliques and generalised theta graphs have played an important part in this work. We give the relationship between the roots of two families of generalised theta graphs.

Delta-matroids and delta-matroid polynomials

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Delta-matroids were introduced and studied by several authors, notably André Bouchet, in the late 1980s. Informally, they are to ribbon (or fat) graphs, what matroids are to graphs. Recent work on ribbon graphs and unconnected work on principal pivot transforms have motivated a resurgence in interest in delta-matroids. We give a brief introduction to delta-matroids and show that various new results or concepts concerning ribbon graphs, in particular concerning ribbon graph polynomials, generalize to delta-matroids.

Some results and conjectures related to the critical problem

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In 1970, Crapo and Rota introduced the critical problem for matroids to provide a unified framework for a number of problems in extremal combinatorial theory including Tutte's 5-Flow Conjecture and Hadwiger's Conjecture. The critical problem provides a link between graphs and matroids that enriches both subjects. This talk will present some results that exemplify how this link has been exploited.

Which biased graphs are group-labelled graphs?

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A biased graph is a graph with a distinguished set of cycles, called balanced, with the property that no theta subgraph contains exactly two balanced cycles. One may construct biased graphs from group-labelled graphs: if Γ is a group, a Γ-labelled graph is a graph G where every edge is oriented and assigned an element from Γ. Then we obtain a biased graph on G by declaring a cycle to be balanced if the product of the group elements along the cycle is the group identity, where we take the inverse of a group element on an edge traversed backwards. In this talk we give a necessary and sufficient condition for a biased graph to be obtained from a group labelling.

This is joint work with Matt DeVos and Daryl Funk.

The Merino-Welsh Conjecture

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One version of the Merino-Welsh conjecture is that if a loopless and bridgeless graph G has $\alpha(G)$ *acyclic orientations*, $\alpha^*(G)$ *totally cyclic orientations* and $\tau(G)$ *spanning trees*, then

$$\alpha(G) \cdot \alpha^*(G) \geq \tau(G)^2.$$

These three parameters are all evaluations of the Tutte polynomial, because $\alpha(G) = T_G(2,0)$, $\tau(G) = T_G(1,1)$ and $\alpha^*(G) = T_G(0,2)$. Therefore this conjecture can immediately be extended to a matroid conjecture in the usual fashion (i.e. replace G by M), thereby obtaining the Merino-Welsh conjecture for matroids.

$$T_M(2,0) \cdot T_M(0,2) \geq T_M(1,1)^2$$

In this talk, I will discuss various aspects of this conjecture, including its resolution in the affirmative for *series-parallel graphs* (which was joint work with Steve Noble).

Generalized Petersen graphs, Tutte's 5-flow conjecture, Beraha conjecture, and all that

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We consider the Tutte polynomial (alias the Q-state Potts-model partition function) for the the family of non-planar cubic graphs known as the generalized Petersen graphs $G(nk, k)$. They can be regarded as good representatives of the class of generic non-planar strip graphs, so we expect to gain new insights about general properties of the Tutte polynomial for non-planar graphs. We first show how to build the transfer matrix for $G(nk, k)$ with fixed $k \geq 1$. Using this object, we study two different (but related) problems:

(1) The real roots of the flow polynomial $\Phi_{G(nk,k)}(Q)$. We find counter-examples of the conjecture of Haggard–Pierce–Royle by showing that there are infinitely many real flow roots $Q > 5$ within the class $G(nk, k)$. In particular, we compute explicitly the flow polynomial of $G(119, 7)$ which has two real roots $Q > 5$, and we prove that the families $G(6n, 6)$ and $G(7n, 7)$ possess real flow roots that accumulate at $Q = 5$ as $n \to \infty$. Therefore, to prove or disprove Tutte's 5-flow conjecture, one should use purely combinatorial methods, in the same way as for the proof of the 4-color theorem.

(2) The real roots of the chromatic polynomial $P_{G(nk,k)}(Q)$. We find that the role played by the Beraha numbers for planar graphs is now played by the non-negative integers for non-planar graphs. At these integer values of Q, there are massive eigenvalue cancellations, in the same way as the eigenvalue cancellations that happen at the Beraha numbers for planar graphs. This result can be extended to what physicists call the Berker–Kadanoff phase for the Q-state Potts model.

Abstracts of Posters

A Tutte-like polynomial for polymatroids

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The Tutte polynomial for matroids is not directly applicable to polymatroids. For instance, deletion-contraction properties do not hold. We construct a polynomial for polymatroids which behaves similarly to the Tutte polynomial of a matroid, and is in fact an evaluation of Tutte when we restrict to matroids. This polynomial is constructed using lattice point counts in the Minkowski sum of the base polytope of a polymatroid and scaled copies of the standard simplex.

Fourientations and the Tutte polynomial

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A fourientation of a graph is a choice for each edge of the graph whether to orient that edge in either direction, bidirect it, or leave it unoriented. Fixing a total order on the edges and a reference orientation of the graph, we investigate properties of cuts and cycles in fourientations which give trivariate generating functions that are generalized Tutte polynomial evaluations of the form

$$(k+m)^{n-1}(k+l)^g T\left(\frac{\alpha k+\beta l+m}{k+m}, \frac{\gamma k+l+\delta m}{k+l}\right)$$

for $\alpha, \gamma \in \{0,1,2\}$ and $\beta, \delta \in \{0,1\}$. We introduce an intersection lattice of 64 cut-cycle fourientation classes enumerated by generalized Tutte polynomial evaluations of this form. We prove these enumerations using a single deletion-contraction argument and classify axiomatically the set of fourientation classes to which our deletion-contraction argument applies. This work unifies and extends earlier results due to many authors over several decades. We also investigate a parallel story of edge colorings and produce an intersection lattice of 64 cut-cycle classes for 4-edge-colorings (implicitly based on internal and external activity) which give generalized Tutte polynomial evaluations of the form above. We put forward the problem of finding a unified bijection between fourientations and 4-edge colorings which respects all of these cut-cycle classes. We conclude by describing how several of these classes of fourientations relate to various geometric, combinatorial, and algebraic objects including bigraphical arrangements, cycle-cocycle reversal systems, graphic Lawrence ideals, divisors on graphs, and zonotopal algebras.

Evaluating the Tutte Polynomial of a Rooted Graph with Bounded Tree-Width

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A rooted graph is a graph with a fixed source vertex called the root. Gordon and McMahon defined the branching greedoid Tutte polynomial of a rooted graph (more generally a greedoid) $G = (V, E)$ by letting the rank function of a subset of edges $A \subseteq E$ be the maximum number of edges in A which form a rooted subtree. Evaluating this polynomial anywhere in the xy-plane is #P-hard apart from at the point $(1, 1)$ and along the curve $(x - 1)(y - 1) = 1$. If we restrict our class of rooted graphs to be those with bounded tree-width, then we can find a linear-time algorithm to evaluate it at any point. The concept of such algorithm is what will be presented in the poster.

On Postnikov-Shapiro algebras

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A. Postnikov and B. Shapiro constructed several algebras associated to G whose dimensions are equal to the number of either spanning trees or subforests of G, for example following definitions for algebra corresponds to subforests: Given a graph G on n vertices; let as associate commuting variables $\phi_e, e \in G$ to all edges of G let Φ_G^F be the algebra generated by $\{\phi_e : e \in G\}$ with relations $\phi_e^2 = 0$, for any $e \in G$. Take any linear order of vertices of G. For $i = 1, \ldots, n$, Set $X_i = \sum_{e \in G} c_{i,e}\phi_e$, where $c_{i,e} = \pm 1$ for vertices incident to e (for the smaller vertex, $c_{i,e} = 1$, for the bigger vertex, is $c_{i,e} = -1$) and 0 otherwise. Denote by $\mathcal{C}_G^F$ the subalgebra of Φ_G^F generated by $X_1, \ldots, X_n$.

Let $\mathbb{K}$ be some field of characteristic 0. Consider the ideal J_G^F in the ring $\mathbb{K}[x_1, \cdots, x_n]$ generated by $p_I^F = \left(\sum_{i \in I} x_i\right)^{D_I + 1}$, where I ranges over all nonempty subsets of vertices, and D_I is the total number of edges from vertices in I to vertices outside the subset I. Define the algebra $\mathcal{B}_G^F$ as the quotient $\mathbb{K}[x_1, \ldots, x_n]/J_G^F$.

Fix some linear order on the edges of G. Define externally activity of subforest F as the number of edges $e \in G \setminus F$ such that subgraph $F + e$ has a cycle and e is the minimal edge in this cycle in the above linear order.

It is well known that the number of spanning trees and subforests with fixed external activity is independent of the linear order on the set of edges of G. The following theorem was proved by Postnikov and Shapiro: The algebras B_G^F and C_G^F are isomorphic. Their total dimension as vector spaces over $\mathbb{K}$ is equal to the number of subforests in the graph G. The dimension of the k-th graded component of these algebras equals the number of subforests F of G with external activity $|G| - |F| - k$.

In particular, the last part of theorem means that we can compute these dimensions from Tutte polynomial of graph G. I will present generalization of these algebras to t-label forests (we change relations ϕ_e^2 to ϕ_e^{t+1}), as well there is definition of second type. For such algebras we also can compute Hilbert Series (dimension of graded components) using only Tutte polynomial, and even the converse, for sufficiently large t, we can compute Tutte polynomial of graph using only Hilbert Series of such algebra.

In addition, I will present key idea of reconstruction algorithm of the matroid of a graph from algebra B_G^F. And also that, for graphs with the same matroid, the algebras are isomorphic, that in turn means that there is the natural bijection between graphic matroids and Postnikov-Shapiro algebras correspond to subforests.

New results for variants of the Merino-Welsh conjecture

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We present new results for the Merino-Welsh conjecture and its two variants, namely the multiplicative version and the convex version.

Let $a(G)$, $c(G)$, $t(G)$ be the numbers of acyclic orientations, totally cyclic orientations and spanning trees of a graph G respectively. The Merino-Welsh conjecture claims that if G is a 2-connected loopless graph, then $t(G) \leq \max\{a(G), c(G)\}$. The multiplicative version is to show $t(G)^2 \leq a(G)c(G)$, and the convex version claims that the Tutte polynomial is convex on the line segment between the points $(2, 0)$ and $(0, 2)$, on which all the numbers $a(G), c(G)$ and $t(G)$ lie.

We prove that, if G has n vertices and m edges, then
$t(G) \leq a(G)$ if $m \leq 1.29(n - 1)$, and
$t(G) \leq c(G)$ if G is 3-edge-connected and $m \geq 3.58(n - 1)$,
which improve Thomassen's result.

Also, inspired by Noble and Royle's proof for series-parallel graphs, we prove the multiplicative version for graphs of pathwidth at most 3 using a computer search. The convex version holds for minimally 2-edge-connected graphs.

Spanning trees and roots of the chromatic polynomial

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The *chromatic polynomial* $P(G, t)$ of a graph G is a polynomial which counts, for each non-negative integer t, the number of proper t-colourings of G. It is a specialisation of the Tutte polynomial. An interval $I \subseteq \mathbb{R}$ is called *zero-free* for a class of graphs $\mathcal{G}$ if $P(G, t) \neq 0$ for all $G \in \mathcal{G}$ and $t \in I$. It is known [1] that $(-\infty, 0)$, $(0, 1)$, and $(1, 32/27]$ are maximal zero-free intervals for the class of all graphs, but one may find larger zero-free intervals for more restrictive classes. We build on results of Thomassen [2] to show that for graphs containing a spanning tree with at most three leaves, the interval $(1, t']$ is maximally zero-free, where $t' \approx 1.2904$ is the smallest real root of the polynomial $(t-2)^6+4(t-1)^2(t-2)^3-(t-1)^4$. We show how an extension of this result could lead to a larger zero-free interval for 3-connected planar graphs.

References.

[1] B. Jackson, A zero-free interval for chromatic polynomials of graphs, *Combin. Probab. Comput.* **2**, (1993) 325-336.
[2] C. Thomassen, Chromatic roots and Hamiltonian paths, *J. Combin. Theory Ser. B* **80**, (2000) 218-224.

Equivalence of Embedded Graphs and Twisted Duality in Hypermaps

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We consider three operations on an edge of a hypermap: giving a half-twist to the edge, taking the partial dual with respect to the edge, and permuting the cyclic order of the edge. We define two hypermaps G and H as *twisted duals* if one can be obtained from the other by using any combination of these local operations. We use $\mathrm{Orb}(H)$ to denote the set of twisted duals of a hypermap H. We show that

$$\mathrm{Orb}(G) = \{H \mid H_m \cong G_m\}.$$

That is, the set of twisted duals of G is precisely the set of all graphs with medial graphs isomorphic (as abstract graphs) to G_m the medial graph of G. This result solves an embedding problem as it describes all of the ways that a given Eulerian graph can be obtained as a medial graph.