Graph Fourientations and the Tutte Polynomial

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Chip-firing

chip-firing

- A chip configuration is a collection of poker chips sitting at the vertices. In keeping with algebraic geometry we may call chip configurations divisors.
- A vertex fires by sending a chip to each of its neighbors.

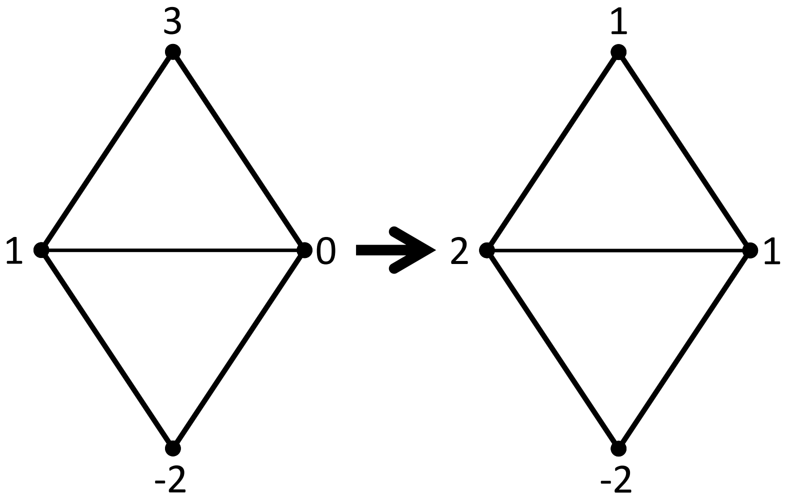

Figure : An example of a chip-firing move

Baker and Norine applied chip-firing to demonstrate a combinatorial Riemann-Roch theorem for graphs analogous to the classical statement for algebraic curves. This result has been utilized for proving theorems in algebraic geometry and number theory.

The Riemann-Roch theorem for graphs [Baker and Norine 07]

$$r(D) - r(K - D) = \deg(D) - g + 1$$

.

Remark

This theorem is secretly about tropical geometry.

Idea

Give a description of chip-firing using partial graph orientations and apply this interpretation to gain insight into the Riemann-Roch formula.

History

- Mosesian observed that if you have an acyclic orientation of a graph, you can reverse the edges at a sink to obtain a new acylcic orientation.
- Björner, Lovász, and Shor noticed that the indegree sequences of the two acyclic orientations are related by firing the sink in question.
- Gioan generalized this setup to arbitrarily full orientations using cut (cocycle) reversals and dual cycle reversals.

Partial graph orientations

- A partial orientation $\mathcal{O}$ of a graph G is an orientation of some edges of G.
- Given an orientation, we associate a chip configuration $D_{\mathcal{O}}$ given by the indegree **-1** of each vertex in $\mathcal{O}$.
- We say that two partial orientations $\mathcal{O}$ and $\mathcal{O}'$ are equivalent in the generalized cycle-cocycle reversal system , written $\mathcal{O} \sim \mathcal{O}'$, if they are related by a sequence of cut reversals, cycle reversals, and edge pivots.

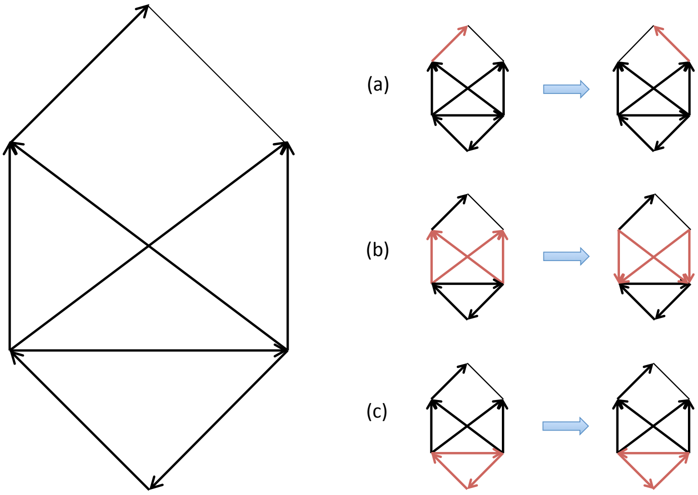

Figure : A partial orientation with (a) an edge pivot, (b) a cocycle reversal, and (c) a cycle reversal.

Theorem [B.]

$\mathcal{O}_1 \sim \mathcal{O}_2$ if and only if $D_{\mathcal{O}_1} \sim D_{\mathcal{O}_2}$.

Theorem [B.]

Given a partial orientation $\mathcal{O}$, either

1. $\mathcal{O} \sim \mathcal{O}'$ where $\mathcal{O}'$ is sourceless $(r(D_{\mathcal{O}}) \geq 0)$ or
2. $\mathcal{O} \sim \mathcal{O}'$ where $\mathcal{O}'$ is acyclic $(r(D_{\mathcal{O}}) = -1)$.

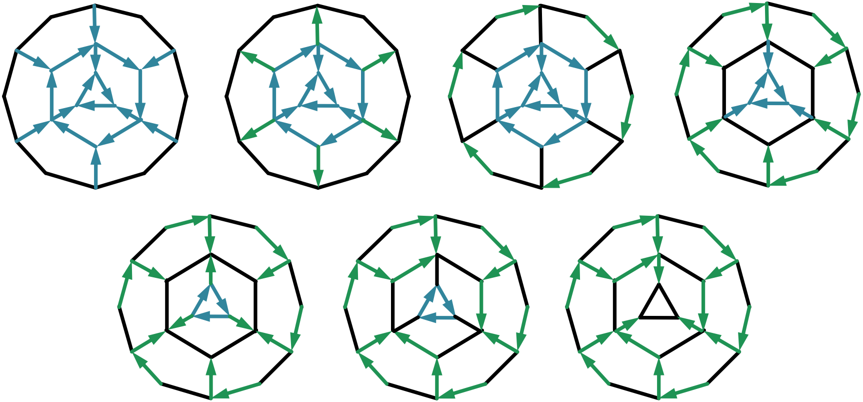

Figure : The unfurling algorithm

Stanley famously proved that the # acyclic full orientations $= T(2,0)$. Can we prove that acyclic partial orientations are also counted by an evaluation of the Tutte polynomial?

Extending acyclic partial orientations

Any acyclic partial orientation $\mathcal{O}$ can be extended to a full acyclic orientation.

Proof: Given an unoriented edge e in $\mathcal{O}$, at least one orientation of e preserves acyclicity. Suppose not, then there is a path from u to v and a path from v to u, hence a cycle was already present in $\mathcal{O}$, a contradiction.

Moreover, both orientations of e preserve acyclicity if and only if $\mathcal{O}/e$ doesn't contain a directed cycle.

Let $f(G)$ be the number of acyclic partial orientations of G.

$$f(G) = 2f(G \setminus e) + f(G/e)$$

$f(bridge) = 3$ and $f(loop) = 1$

Theorem [Gessel-Sagan]

$2^g T(3, 1/2) = \#$ acyclic partial orientations

If we allow each oriented edge to take one of k colors and each unoriented edge to have one of l colors, then we find that

$$f(G) = k^{n-1}(k+l)^g T(2k+l, \frac{l}{k+l}).$$

In what follows, we will work with the additional data of a reference orientation $\mathcal{O}_{ref}$ of G and a total order $<$ on the edges.

Some more generalized Tutte polynomial evaluations [B.]

- $2^{n-1}T(1/2,3) = \#$ strongly connected partial orientations
- $2^g T(3,1) = \#$ partial orientations modulo directed cycle reversals $=$ $\#$ partial orientations such that the minimum edge in each directed cycle reversals is oriented in the same direction as in $\mathcal{O}_{ref}$
- $2^{n-1}T(1,3) = \#$ partial orientations modulo cut reversals $=$ $\#$ partial orientations such that the minimum edge in each directed cycle reversals is oriented in the same direction as in $\mathcal{O}_{ref}$

Remark on combinatorial commutative algebra

The above results are related to graphic and cographic Lawrence ideals and a theorem of Sturmfels, which states that a minimal binomial generating set for a Lawrence ideal is a universal Gröbner basis.

Bigraphical Arrangements

In the process of proving the G-Shi conjecture, Sam Hopkins and Dave Perkinson introdcuded and investigated bigraphical arrangements.
See Sam's poster for more about this interesting story.

Given an edge $e = (v_i, v_j)$ in G, we take two hyperplanes $x_i - x_j = a_{i,j}$ and $x_i - x_j = a_{j,i}$.

Theorem [Hopkins - Perkinson]

- $2^{n-1}T(3/2, 1) = \#$ of regions in a generic bigraphical arrangement. (The regions are in bijection with divisors associated to certain acyclic partial orientations).
- $2^{n-1}T(1/2, 1) = \#$ of bounded regions in a generic bigraphical arrangement.

Sam and I began working together to see if we could find a common framework for viewing these results.

Bigraphical Arrangements

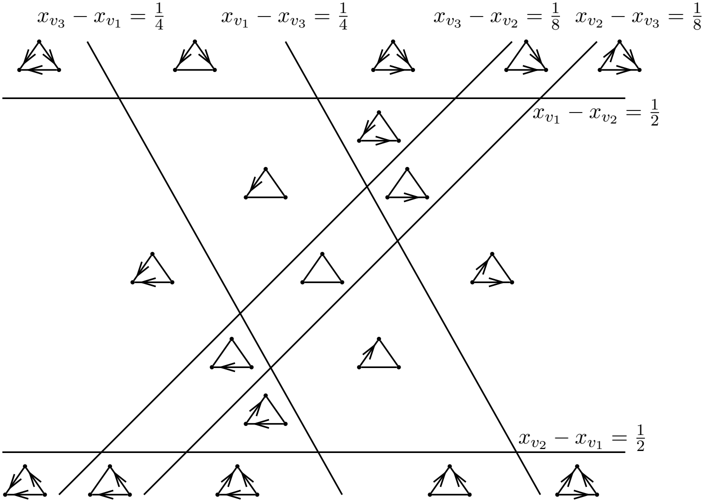

Figure : An example of a bigraphical arrangement with exponential parameters

Fourientations

A fourientation is a choice for each edge of the graph whether to orient that edge in either direction, bidirect it, or leave it unoriented. There are $4^{|E|}$ fourientations of a given graph.

We say that a fourientation is q-connected, or rooted, if there is a directed path from q to every other vertex.

More results of Gessel and Sagan

- $2^g T(1, 1/2) = \#$ of q-connected acyclic partial orientations.
- $2^{|E|} T(1, 2) = \#$ of q-connected fourientations.

Remark: The divisors associated to the acyclic q-connected partial orientations are the G-parking functions which are dual to the recurrent configurations from the sandpile model. Merino famously proved that these are enumerated by $y^g T(1, 1/y)$.

Fourientation Properties

Let e^+ and e^- denote the orientations of an edge which agree and disagree with $\mathcal{O}_{ref}$, repsectively.
Let $e^{\emptyset}$ and $e^{\pm}$ denote the unoriented and bidirected e, respectively.

Definition

- A potential cut (cycle) in a fourientation is obtained from a directed cut (cycle) by unorienting (bidirecting) some of the edges.
- Let $X \subset \{\emptyset, +, -\}(\{+, -, \{+, -\}\})$. We say that a cut (cycle) in $\mathcal{O}$ is bad with respect to X if the minimum edge e in this cut is of the form e^{δ} for $\delta \in X$ and if $\delta = \emptyset$ $(\{+, -\})$, then the cut is oriented as e^-.
- A fourientation $\mathcal{O}$ is good with respect to X if none of its potential cuts and cycles are bad.

Some fourientation terminology

- **Cut general** ($X = \emptyset$)
- **Cut directed** ($X = \{\emptyset\}$)
- **Cut neutral** ($X = \{\{-\}\}$)
- **Cut coneutral** ($X = \{\{+\}\}$)
- **Cut connected** ($X = \{\emptyset, \{-\}\}$)
- **Cut coconnected** ($X = \{\emptyset, \{+\}\}$)
- **Cut positive** ($X = \{\{-\}, \{+\}\}$)
- **Cut free** ($X = \{\emptyset, \{-\}, \{+\}\}$)

Definition

A (k,l,m)-fourientation is obtained from a fourientation by assigning each of the oriented edges one of k colors, each of the unoriented edges one of l colors, and each of the bidirected edges one of m colors.

Theorem [B., Hopkins]

The number of good (k, l, m)-fourientations defined by a Tutte cut-cycle property is

$$(k+m)^{n-1}(k+l)^g T\left(\frac{x_0}{k+m}, \frac{y_0}{k+l}\right)$$

where

$$x_0 = \begin{cases} 2k+l+m & \text{if cut general} \\ 2k+m & \text{if cut directed or codirected} \\ k+l+m & \text{if cut neutral} \\ k+m & \text{if cut connected or coconected} \\ l+m & \text{if cut positive} \\ m & \text{if cut free} \end{cases}$$

Some remarks

Remarks

- We prove our result by a single uniform deletion-contraction argument.
- Moreover, we prove that our lattice of fourientation classes is exhaustive in a precise sense.
- We recover results about q-connectedness by taking a reference orientation such that the min edge in any cut is oriented away from q. This is further reducable to "tree data".
- We have an analgous lattice of 4 edge colorings.
- We conjecture a universal bijection between fourientations and 4 edge colorings which respects all of our cut-cycle classes.
- Replacing the two k's in our formula with k_1 and k_2 is equivalent to the existence of such a bijection.

Fourientations – *rygb-edge-colorings*

	General	Cut pos./neg. Cut directed	Cut open Cut (co)-con.	Cut free	
General	$2^{\lvert E\rvert}T(2,2)$	$2^{\lvert E\rvert}T(\frac{3}{2},2)$	$2^{\lvert E\rvert}T(1,2)$	$2^{\lvert E\rvert}T(\frac{1}{2},\frac{3}{2})$	*General*
Cycle pos./neg. Cycle directed	$2^{\lvert E\rvert}T(2,\frac{3}{2})$	$2^{\lvert E\rvert}T(\frac{3}{2},\frac{3}{2})$	$2^{\lvert E\rvert}T(1,\frac{3}{2})$	$2^{\lvert E\rvert}T(\frac{1}{2},\frac{3}{2})$	*r/y-pseudo* *ry forest* *b internal*
Cycle open Cycle (co)-con.	$2^{\lvert E\rvert}T(2,1)$	$2^{\lvert E\rvert}T(\frac{3}{2},1)$	$2^{\lvert E\rvert}T(1,1)$	$2^{\lvert E\rvert}T(\frac{1}{2},1)$	*r/y-pseudo* *ry forest* *+ b internal* *ry forest*
Cycle free	$2^{\lvert E\rvert}T(2,\frac{1}{2})$	$2^{\lvert E\rvert}T(\frac{3}{2},\frac{1}{2})$	$2^{\lvert E\rvert}T(1,\frac{1}{2})$	$2^{\lvert E\rvert}T(\frac{1}{2},\frac{1}{2})$	*ry forest* *+ b internal*
	General	*b/g-pseudo* *ry spanning* *r external*	*b/g-pseudo* *ry spanning* *+ r external* *ry spanning*	*ry spanning* *+ r external*	

Type A classes of partial orientations
rgb-edge-colorings

	General	Cut pos./neg. Cut dir.	Cut open Cut (co)-con.	Cut free	
Gen.	2^g $T(3,\frac{3}{2})$	2^g $T(2,\frac{3}{2})$	2^g $T(1,\frac{3}{2})$	2^g $T(0,\frac{3}{2})$	*Gen.*
Cycle min. Cycle max.	2^g $T(3,1)$	2^g $T(2,1)$	2^g $T(1,1)$	2^g $T(0,1)$	*r forest* *b int.*
Acyc.	2^g $T(3,\frac{1}{2})$	2^g $T(2,\frac{1}{2})$	2^g $T(1,\frac{1}{2})$	2^g $T(0,\frac{1}{2})$	*r forest* *+ b int.*
	General	*b/g-pseudo* *ry span.* *r ext.*	*b/g-pseudo* *ry span.* *+ r ext.* *ry span.*	*r span.* *+ r ext.*	

Type B classes of partial orientations
ryb-edge-colorings

	General	Cut min./max.	Strong. con.	
Gen.	2^{n-1} $T(\frac{3}{2},3)$	2^{n-1} $T(1,3)$	2^{n-1} $T(\frac{1}{2},3)$	*Gen.*
Cycle pos./neg. Cycle dir.	2^{n-1} $T(\frac{3}{2},2)$	2^{n-1} $T(1,2)$	2^{n-1} $T(\frac{1}{2},2)$	*r/y-pseudo* *ry forest* *b int.*
Cycle open Cycle (co)-con.	2^{n-1} $T(\frac{3}{2},1)$	2^{n-1} $T(1,1)$	2^{n-1} $T(\frac{1}{2},1)$	*r/y-pseudo* *ry forest* *+ b int.* *ry forest*
Cycle free	2^{n-1} $T(\frac{3}{2},0)$	2^{n-1} $T(1,0)$	2^{n-1} $T(\frac{1}{2},0)$	*ry forest* *+ b int.*
	General	*ry span.* *r ext.*	*ry span.* *+ r ext.*	

Total orientations – *Subgraphs (rb-edge-colorings)*

	General	Cut min./max.	Strongly connected	
General	$T(2,2)$	$T(1,2)$	$T(0,2)$	*General*
Cycle min./max.	$T(2,1)$	$T(1,1)$	$T(0,1)$	*Forest* *Internal*
Acylic	$T(2,0)$	$T(1,0)$	$T(0,0)$	*Forest +* *Internal*
	General	*Spanning* *Internal*	*Spanning* *+ External*	

Figure 1. Four tables showing how the various classes of orientations, and their associated edge colorings, are enumerated by generalized Tutte polynomial evaluations.

Notice at the bottom we recover a 3×3 table of (mostly classical) enumerations of full orientations due to several authors which were unified by Gioan and Bernardi.

Orientations and the Tutte Polynomial $T(x, y)$ for $0 \leq x, y \leq 2$

- T(2,2) $= 2^{|E|} = \#$ orientations
- T(2,0) $= \#$ acyclic orientations (Stanley)
- T(1,0) $= \#$ q-connected acyclic orientations (Greene - Zaslavsky)
- T(0,2) $= \#$ strongly connected orientations (Las Vergnas)
- T(0,1) $= \#$ "cycle minimal" strongly connected orientations. (Greene - Zaslavsky)
- T(2,1) $=$ indegree sequences of orientations (Stanley)
- T(1,2) $=$ rooted orientations (Gioan)
- T(1,1) $=$ indegree sequences of rooted orientations (Gioan)
- T(0,0) $= 0$