Spanning Trees and Roots of the Chromatic Polynomial

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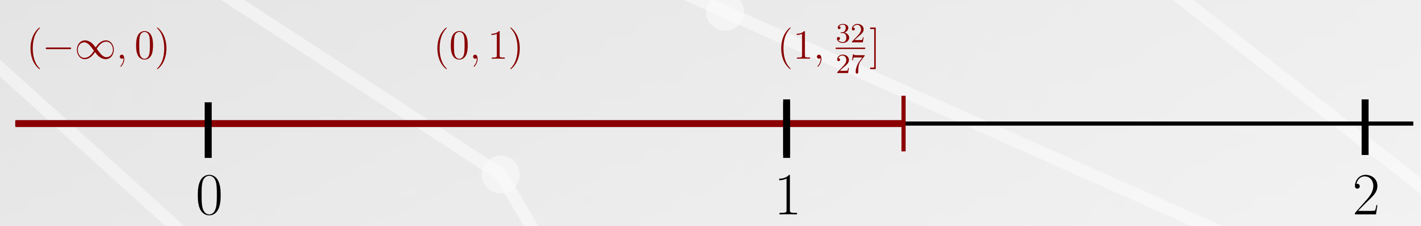
Introduction

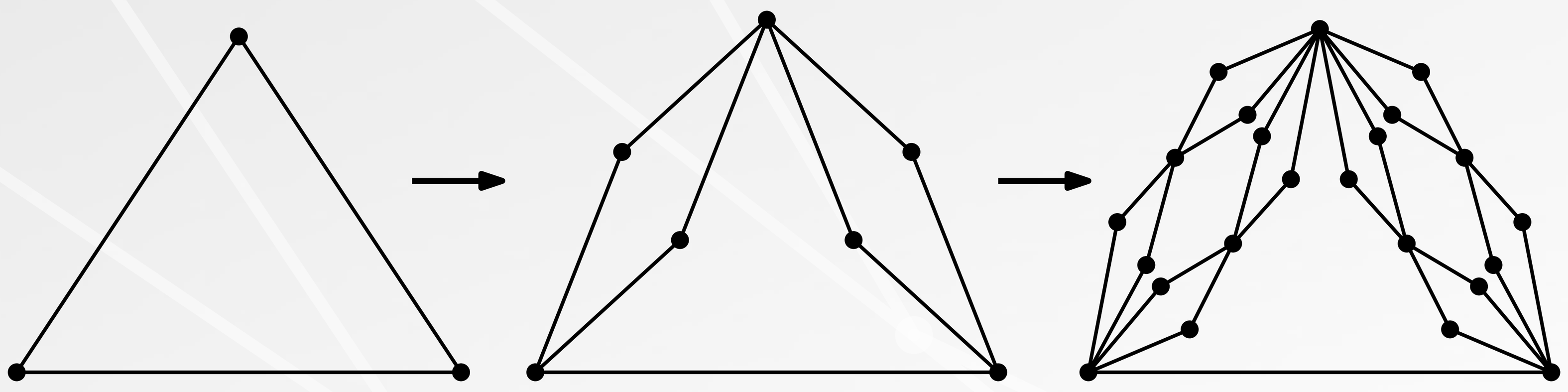
To each graph G, we can associate a polynomial $P(G,t)$ which counts, for each non-negative integer t, the number of proper t-colourings of G.

Definition. *An interval $I \subseteq \mathbb{R}$ is* **zero-free** *for a class of graphs $\mathcal{G}$ if $P(G,t) \neq 0$ for all $G \in \mathcal{G}$ and all $t \in I$.*

Theorem. [2] *For all graphs the following intervals are maximally zero-free:*



The following sequence of graphs have chromatic roots converging to $32/27$.



Problem. *Is $(1, 32/27+\varepsilon]$ zero-free for the class of 3-connected graphs?*

Barnette proved that a 3-connected **planar** graph has a spanning tree of maximum degree 3 so the following is interesting.

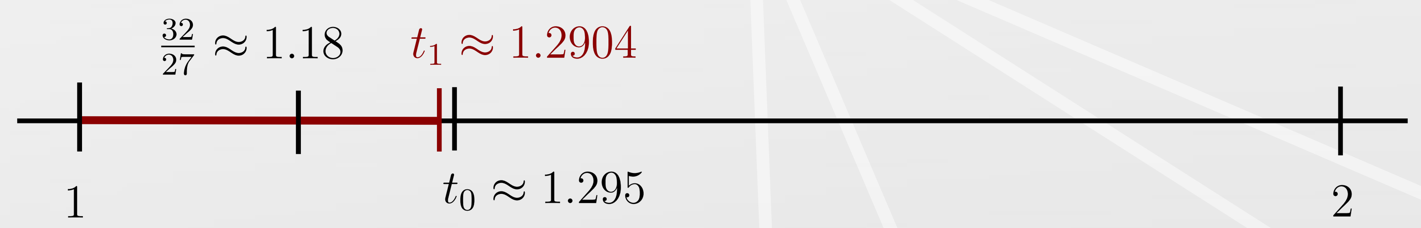
Conjecture. *There is $\varepsilon > 0$ such that $(1, 32/27+\varepsilon]$ is zero-free for the class of graphs with a spanning tree of* **maximum degree 3**.

The following is known.

Theorem. [3] *The interval $(1, t_0]$ is maximally zero-free for the class of graphs with a* **Hamiltonian path**, *where $t_0 \approx 1.295$ is the unique real root of the polynomial $(t-2)^3 + 4(t-1)^2$.*

Main Result

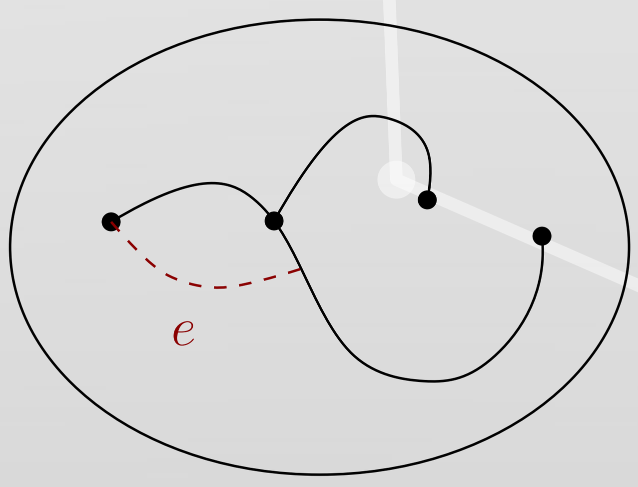
Theorem. *The interval $(1, t_1]$ is maximally zero-free for the class of graphs having a* **spanning tree** *with* **at most three leaves**, *where $t_1 \approx 1.2904$ is the smallest real root of the polynomial $(t-2)^6 + 4(t-1)^2(t-2)^3 - (t-1)^4$.*



Method of Proof

We prove the stronger statement:

If G is **2-connected** and has a spanning tree with at most three leaves, then $(-1)^n P(G,t) > 0$ for $t \in (1, t_1]$.



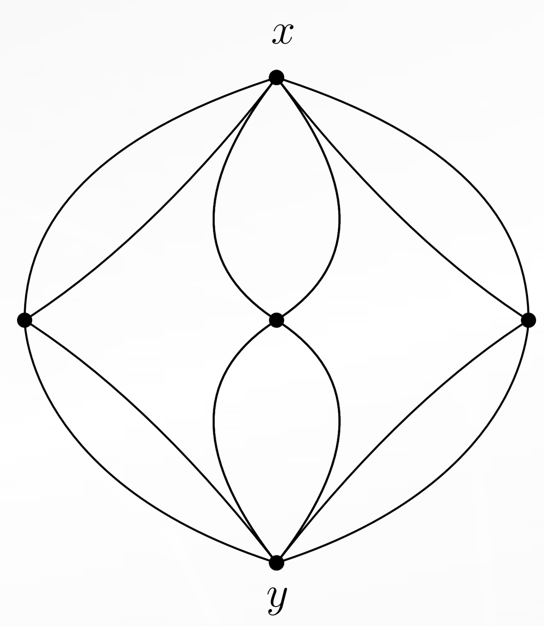
We would like to use **induction** and the **deletion-contraction** identity in the following way.

$$\begin{aligned}(-1)^n P(G,t) &= (-1)^n [P(G-e,t) - P(G/e,t)] \\ &= (-1)^n P(G-e,t) + (-1)^{n-1} P(G/e,t) \\ &> 0.\end{aligned}$$

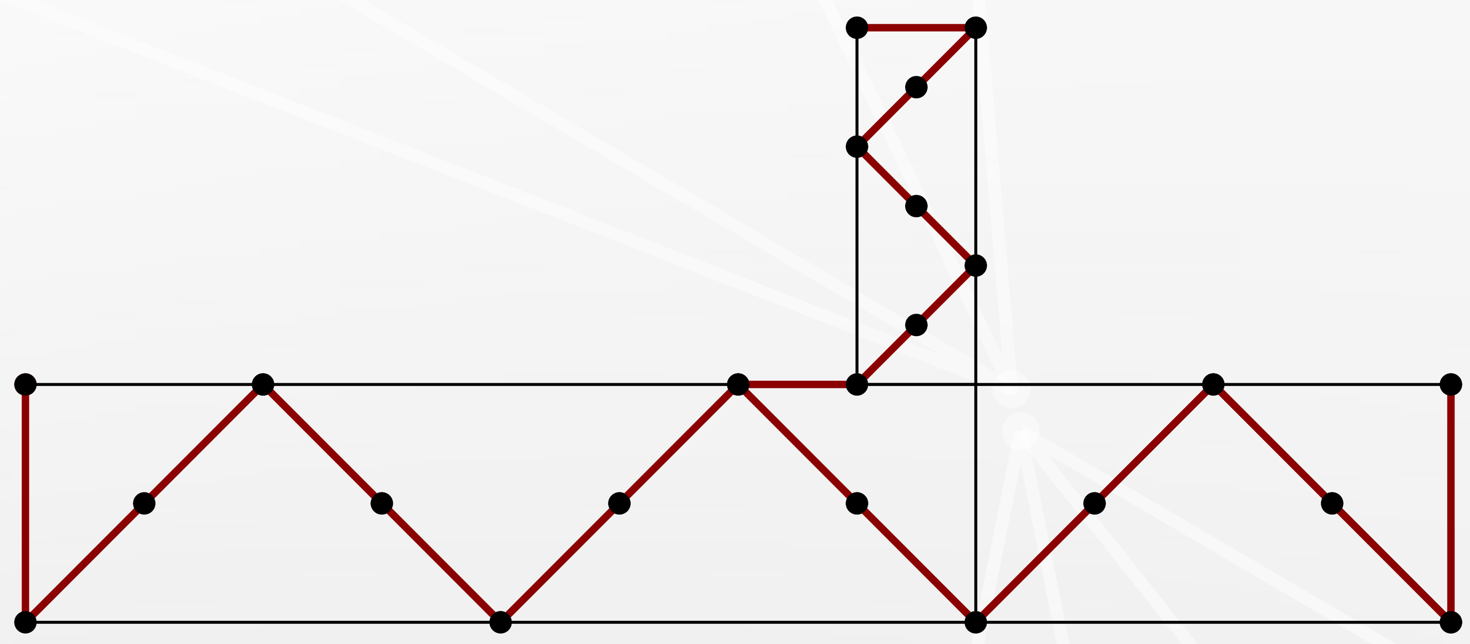
The problem is that $G-e$ and G/e might not be 2-connected. But then a **smallest counter-example** cannot be 3-connected.

In a similar way, using results of Dong and Koh [1], we are able to deduce the following **structure** about a smallest counter-example G.

- G is 2-connected but not 3-connected.
- For every 2-cut $\{x,y\}$, we have $xy \notin E(G)$ and G has precisely three $\{x,y\}$-bridges, none of which is 2-connected.



A graph with this structure and which has a spanning tree with at most three leaves can be transformed to a graph of the following form by a sequence of **Whitney 2-switches**.



For each fixed $t \in (1, t_1]$ we can express the chromatic polynomial of these graphs as

$$P(G_{i,j,k}, t) = A\alpha^j + B\beta^j$$

where A and B depend on i, k and t and

$$\alpha, \beta = \frac{1}{2}\left((t-2)^2 \pm \sqrt{(t-2)^4 + 4(t-1)^2(t-2)}\right).$$

We are able to show that t_1 is the **infimum** of the non-trivial roots of all $P(G_{i,j,k}, t)$ for $i,j,k \in \mathbb{N}$, where t_1 is the smallest root of the polynomial

$$(t-2)^6 + 4(t-1)^2(t-2)^3 - (t-1)^4.$$

References and Acknowledgements

[1] F. M. Dong and K. M. Koh, Domination numbers and zeros of chromatic polynomials, *Discrete Mathematics* **308** (2008) 1930-1940

[2] B. Jackson, A zero-free interval for chromatic polynomials of graphs, *Combin. Probab. Comput.* **2**, (1993) 325-336.

[3] C. Thomassen, Chromatic roots and Hamiltonian paths, *J. Combin. Theory Ser. B* **80**, (2000) 218-224.

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