Generalized Petersen graphs, Tutte's 5-flow conjecture, Beraha conjecture, and all that

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Workshop on New Directions for the Tutte Polynomial

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Thanks to

- Jesper L. Jacobsen (LPTENS)

  Related publications:

  - Is the five-flow conjecture almost false? J. Combin. Theory B **103** (2013) 532-565.
  - A generalized Beraha conjecture for non-planar graphs. Nucl. Phys. B **875** (2013) 678-718.

- Gordon Royle (Western Australia).
- Alan D. Sokal (NYU/UCL).

Main objects

- Tutte polynomial of a *general* graph $G = (V, E)$ (loops and multi-edges allowed) in the Fortuin–Kasteleyn formulation:

$$Z_G(Q, v) \;=\; \sum_{A \subseteq E} Q^{k(A)}\, v^{|A|} \;=\; (x-1)^{k(E)}\, (y-1)^{|V|}\, T_G(x, y)$$

  - $k(A)$ is the number of connected components of (V, A).
  - $Z_G(Q, v)$ is the partition function of the Q-state Potts model at temperature-like variable $v = e^{J/T} - 1$ on G.
  - $x \;=\; 1 + Q/v$, and $y \;=\; v + 1$.

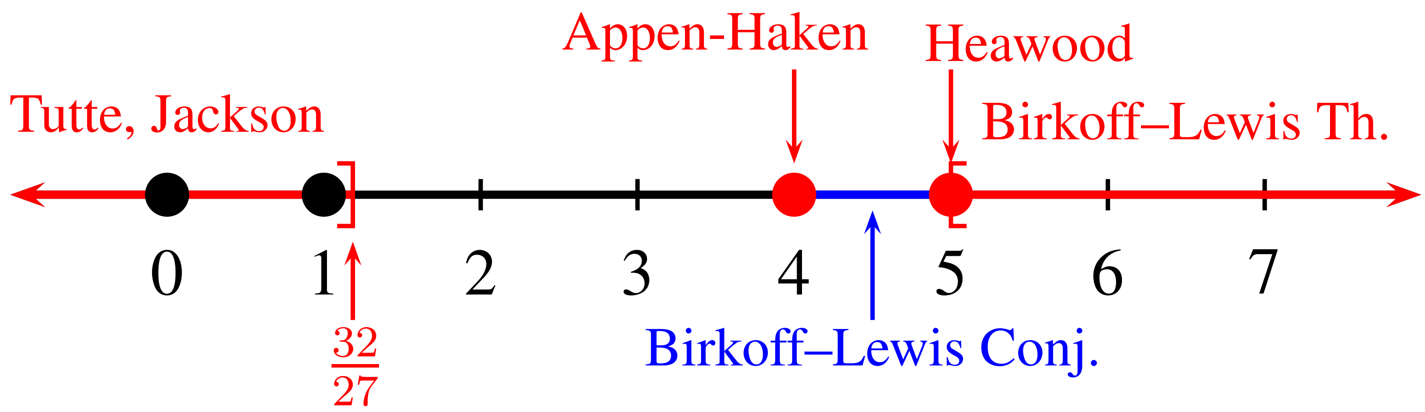
- Chromatic polynomial of a simple graph: $P_G(Q) = Z_G(Q, -1)$.
- Flow polynomial of a bridgeless graph:

$$\Phi_G(Q) = (-1)^{|E|}\, Q^{-|V|}\, Z_G(Q, -Q).$$

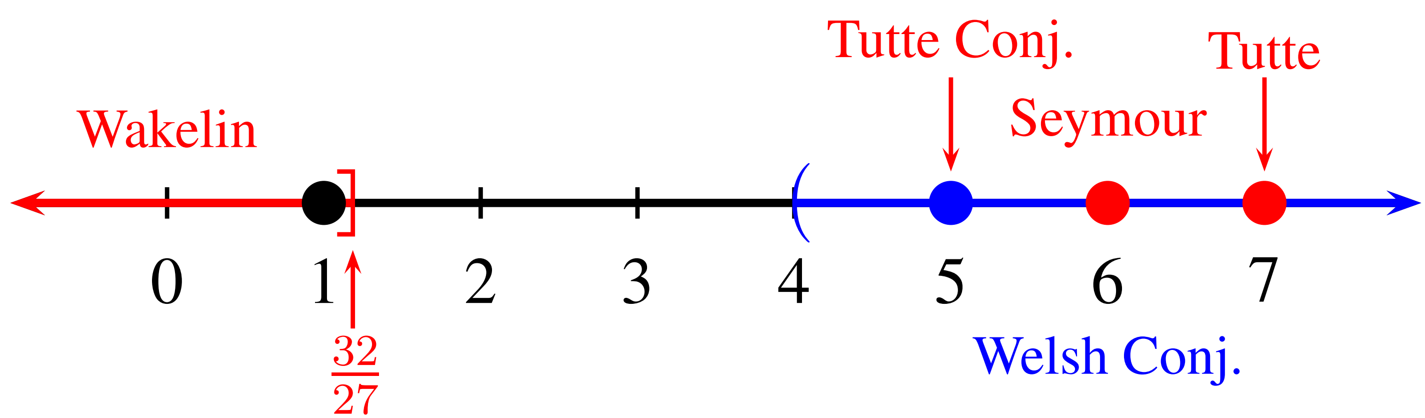
- If G is planar, (G, G^*) is a dual pair, and $P_{G^*}(Q) \;=\; Q\, \Phi_G(Q)$.

Real zero-free intervals for P_G and Φ_G

- P_G for planar graphs G: $P_G(Q) > 0$ for $Q \geq Q_0$ with $4 \leq Q_0 \leq 5$:



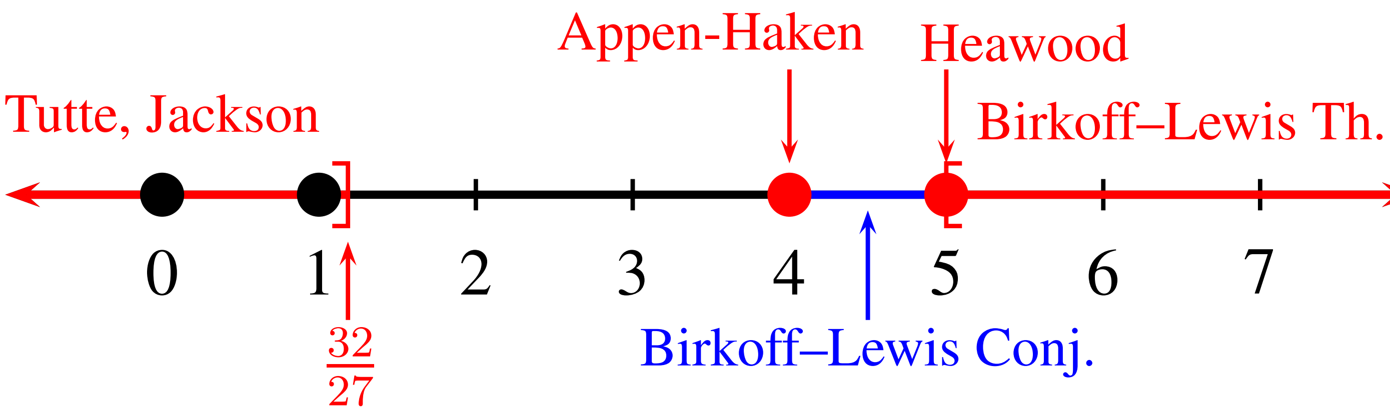
- Φ_G for bridgeless graphs G: $Q \to Q$ scenario = "planar duality"



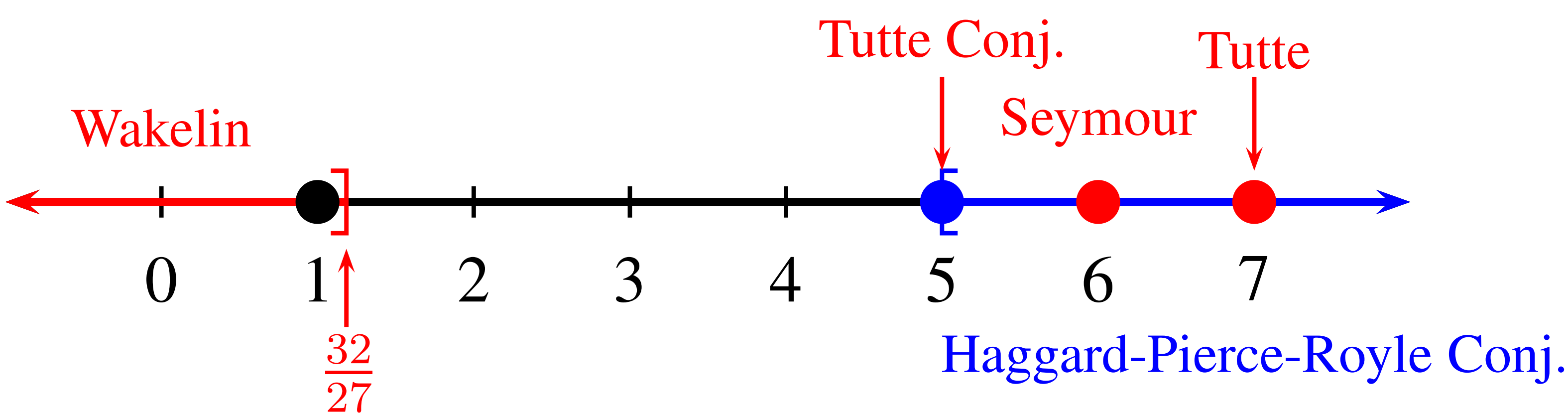
- $\Phi_{G(5,2)}(4) = 0$, where $G(5,2)$ = Petersen graph.
- Welsh's conjecture is false: $\Phi_{G(16,6)}(Q) = 0$ at $Q \approx 4.025, 4.233 > 4$ (Haggard-Pierce-Royle, 2010).

Real zero-free intervals for P_G and Φ_G (2)

- P_G for planar graphs G: $P_G(Q) > 0$ for $Q \geq Q_0$ with $4 \leq Q_0 \leq 5$:



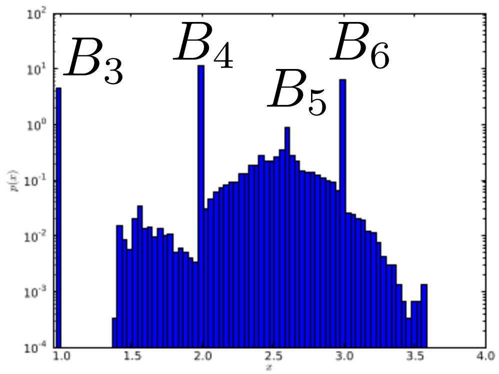
- Φ_G for bridgeless graphs G: $Q \to Q + 1$ scenario



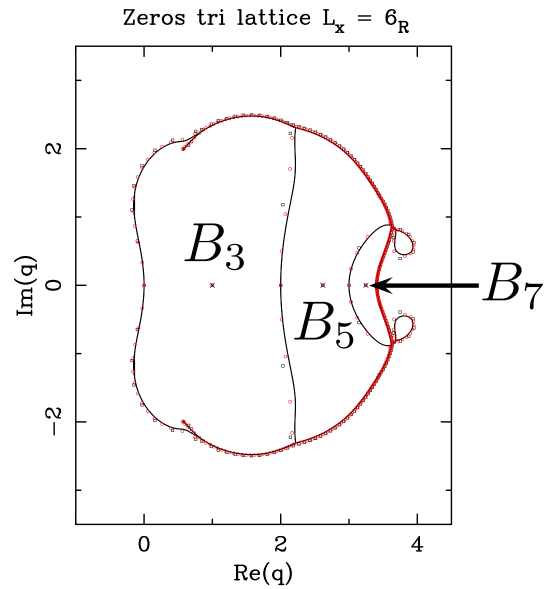
- Open question: Is HPR's conjecture true?

Accumulation points of real chromatic roots

- Real chromatic roots for planar graphs seem to accumulate around the Beraha numbers $B_n = 4\cos(\pi/n)^2$, with integer $n \geq 2$.



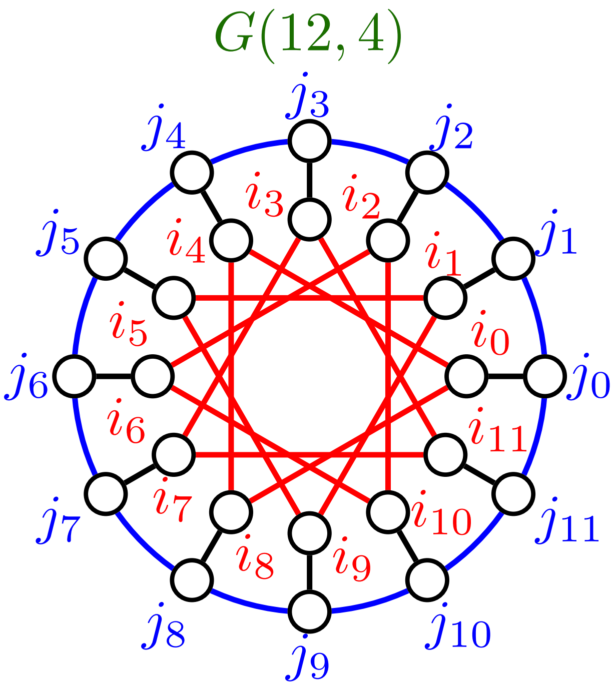
Bedini–Jacobsen (2010)



Jacobsen–JS (2006)

- Beraha conjecture: The Beraha numbers B_n are accumulation points for the real chromatic roots of families of planar recursive graphs. (Warning: there are several versions of this conjecture!)
- Open question: What happens for non-planar G?

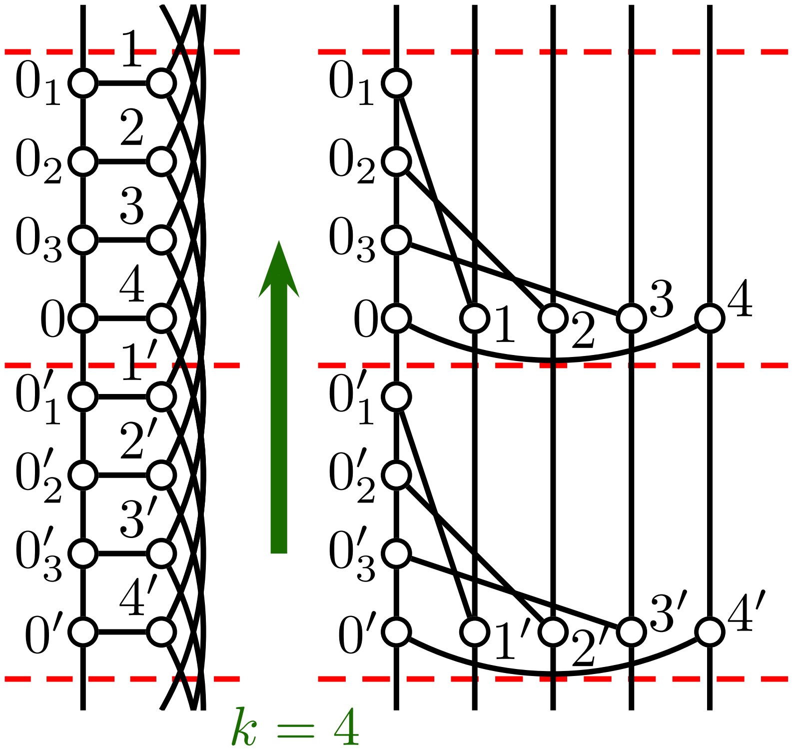
Generalized Petersen graphs $G(m,k)$ (Watkins, 1969)



- $G(m,k) = (V_{m,k}, E_{m,k})$
- $V_{m,k} = \{i_0, i_1, \ldots, i_{m-1}, j_0, j_1, \ldots, j_{m-1}\}$.
- $E_{m,k} = E_i \cup E_j \cup E_{i,j}$.
- $E_j = \{\{j_s, j_{(s+1 \bmod m)}\} \colon 0 \le s \le m-1\}$.
- $E_{i,j} = \{\{i_s, j_s\} \colon 0 \le s \le m-1\}$.
- $E_i = \{\{i_s, i_{(s+k \bmod m)}\} \colon 0 \le s \le m-1\}$.

- $G(m,k)$ are cubic, recursive with large girth (= 8 for $m, k \gg 1$).
- $G(m,k)$ are non-planar (except $(m,k) = (3,2), (p,1), (2p,2)$).
- Good candidates to study these open questions:
  - Large real flow roots seem to be related to large girth values.
  - They are good representatives of the class of generic non-planar strip graphs (TM structure).

Transfer-matrix approach to $G(nk, k)$

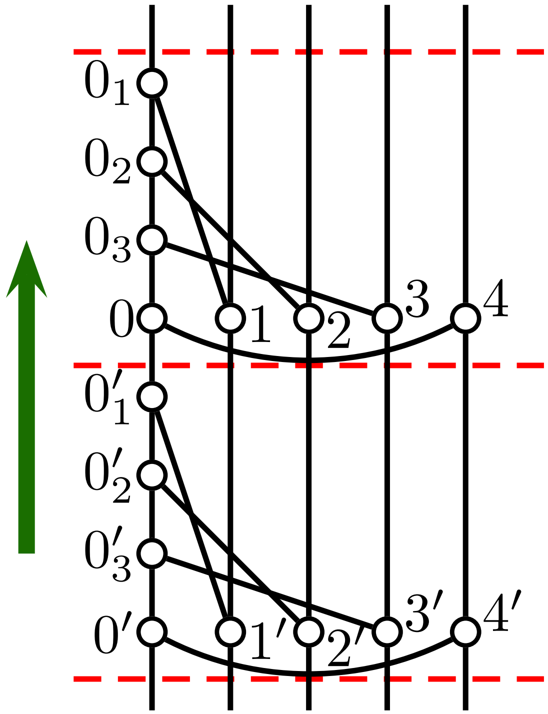


- n layers with $2k$ vertices/layer.
- Cyclic bc's in the transfer direction ↑.
- A "smart" re-drawing is a *must*!
- Layer "effective" width $L = k + 1$.
- Goal: build a matrix T_L such that

$$Z_{G(nk,k)} = \boldsymbol{u}^T \cdot (\mathsf{T}_L)^n \cdot \boldsymbol{v}_{\rm id}.$$

- T_L acts on the space of partitions $\mathcal{P}$ of $\{0', 1', \ldots, k', 0, 1, \ldots, k\}$.
- We start from $\boldsymbol{v}_{\rm id} = \{\{0, 0'\}, \ldots, \{k, k'\}\}$, and T_L adds a layer.
- If $\boldsymbol{v}_\mathcal{P}$ is associated to $\mathcal{P}$, $\boldsymbol{v}_{\mathcal{P}'} = \prod_{i=0}^{k} \mathsf{J}_{i,i'} \cdot \boldsymbol{v}_\mathcal{P}$ with $\mathsf{J}_{i,j}$ being the join operator that amalgamates the blocks containing i, j.
- $\boldsymbol{u}^T \cdot \boldsymbol{v}_\mathcal{P} = Q^{|\mathcal{P}'|}$, with $|\mathcal{P}'| =$ # of blocks in $\boldsymbol{v}_{\mathcal{P}'}$.

Transfer-matrix approach to $G(nk, k)$ (2)



- $Z_{G(nk,k)} = \boldsymbol{u}^T \cdot (\mathsf{T}_L)^n \cdot \boldsymbol{v}_{\rm id}$.
- Sparse-matrix decomposition of T_L in terms of *horizontal* $\mathsf{H}_{i,j} = I + v\, \mathsf{J}_{i,j}$ and *vertical* $\mathsf{V}_i = v\, I + \mathsf{D}_i$ operators.
- D_i is a detach operator: removes site i from its block and turn it into a *singleton* (if i was already a singleton, it gives a factor Q).

- $\mathsf{T}_L = \mathsf{H}_{0,1} \cdot \mathsf{V}_0 \cdot \mathsf{H}_{0,2} \cdot \mathsf{V}_0 \cdot \mathsf{H}_{0,3} \cdots \mathsf{V}_0 \cdot \mathsf{H}_{0,k} \cdot \mathsf{V}_k \cdot \mathsf{V}_{k-1} \cdots \mathsf{V}_1 \cdot \mathsf{V}_0$.
- $Z_{G(nk,k)}(Q,v) = \sum_{j=0}^{\dim \mathsf{T}_L} \alpha_j(Q,v)\, \mu_j(Q,v)^n$.
- But $\dim \mathsf{T}_L = B(2(k+1))$. The Bell numbers $B(n)$ grow very fast with n: $\log B(n) \sim n \log n \left[1 - \frac{\log(\log n)}{\log n} + O\left(\frac{1}{\log n}\right)\right]$.

How to compute the symbolic T_L efficiently?

- T_L does *not* act on the bottom row $\Rightarrow$ T_L has a block-diagonal form such that each diagonal block corresponds to a given bottom-row partition. All of them give the same eigenvalues, so we choose a simple one.
- A link is a block in a partition $\mathcal{P}$ that connects the top and bottom rows, e.g. $\mathcal{P} = \{\{0', 1', 0\}, \{2'\}, \{1, 2\}\}$ with $\ell = 1$.
- Both H_{ij} and V_i can't increase the number of links ℓ $\Rightarrow$ If we order the partitions appropriately, T_L has an upper-triangular block form, where each diagonal block $\mathsf{T}_{L,\ell}$ corresponds to a fixed number of links $0 \leq \ell \leq L = k + 1$.
- Given a *fixed* number of links ℓ, they can be interchanged in every possible way $\Rightarrow$ the relevant group is S_ℓ.

How to compute the symbolic T_L efficiently? (2)

- Halverson–Ram, *Partition algebras*, Eur. J.C. **26** (2005) 689-921:
  - The relevant eigenvalues of $\mathsf{T}_{L,\ell}$ come from the irreducible representations $\lambda \in S_\ell$. If we choose the appropriate basis, then $\mathsf{T}_{L,\ell}$ takes a block-diagonal form, where each diagonal block $\mathsf{T}_{L,\ell,\lambda}$ corresponds to a different irreducible representation λ.
  - Then, the Tutte polynomial is:

$$
\begin{aligned}
Z_{G(nk,k)}(Q,v) &= \sum_{\ell=0}^{k+1} \sum_{\lambda \in S_\ell} \alpha_{\ell,\lambda}(Q)\, \mathrm{tr}\,(\mathsf{T}_{L,\ell,\lambda})^n\,, \\
\alpha_{\ell,\lambda}(Q) &= \frac{\dim\lambda}{\ell!} \prod_{i=0}^{\ell-1} (Q - i - \lambda_{\ell-i})\,.
\end{aligned}
$$

λ is considered through its Young diagram $Y(\lambda) = (\lambda_1, \ldots, \lambda_\ell)$ ($\lambda_i =$ # of boxes in the i-th row).

How to compute the symbolic $Z_{G(nk,k)}$ efficiently?

$$Z_{G(nk,k)}(Q,v) \;=\; \sum_{\ell=0}^{k+1} \sum_{\lambda\in S_\ell} \alpha_{\ell,\lambda}(Q)\ \mathrm{tr}\,(\mathsf{T}_{L,\ell,\lambda})^n \ .$$

- This is true irrespective if some of the eigenvalues are equal or not.
- We look for a "complete" decomposition for $Z_{G(nk,k)}$ in terms of *distinct* eigenvalues. We find for $1 \le k \le 6$:

$$\mathsf{T}_{L,\ell,\lambda} \;=\; \begin{pmatrix} \mathsf{D}_{L,\ell,\lambda} & \mathsf{F}_{L,\ell,\lambda} \\ 0 & \mathsf{T}^{(nt)}_{L,\ell,\lambda} \end{pmatrix} \ .$$

  - $\mathsf{D}_{L,\ell,\lambda}$ is a diagonal matrix with all its diagonal entries are the "trivial" eigenvalue $\mu_{L,k+1} = v^{2k}$. ($\dim \mathsf{D}_{L,0} = 0$.)
  - $\mathsf{T}^{(nt)}_{L,\ell,\lambda}$ is the diagonal block containing the non-trivial eigenvalues $\mu_{L,\ell,\lambda,s}$. ($\dim \mathsf{T}^{(nt)}_{L,k+1,\lambda} = 0$.)
  - $\mathsf{T}^{(nt)}_{L,k,\lambda}$ contains only $2k+1$ non-trivial eigenvalues $\mu_{L,k,s}$ of multiplicity $\dim\lambda$ for all $\lambda \in S_k$. (use $\mathsf{T}^{(nt)}_{L,k}$.)

How to compute the symbolic $Z_{G(nk,k)}$ efficiently? (2)

- For any $1 \le k \le 6$, we have proven that

$$\begin{aligned} Z_{G(nk,k)}(Q,v) &= \sum_{\ell=0}^{k-1} \sum_{\lambda \in S_\ell} \alpha_{\ell,\lambda} \operatorname{tr}(\mathsf{T}^{(nt)}_{L,\ell,\lambda})^n + \beta_k \operatorname{tr}(\mathsf{T}^{(nt)}_{L,k})^n + \gamma_{k+1} v^{2nk} \\ &= \sum_{\ell=0}^{k-1} \sum_{\lambda \in S_\ell} \alpha_{\ell,\lambda} \sum_{s=1}^{N_k(\ell,\lambda)} \mu^n_{L,\ell,\lambda,s} + \beta_k \sum_{s=1}^{2k+1} \mu^n_{L,k,s} + \gamma_{k+1} v^{2nk}, \end{aligned}$$

where $N_k(\ell,\lambda) = \dim \mathsf{T}^{(nt)}_{L,\ell,\lambda}$, and

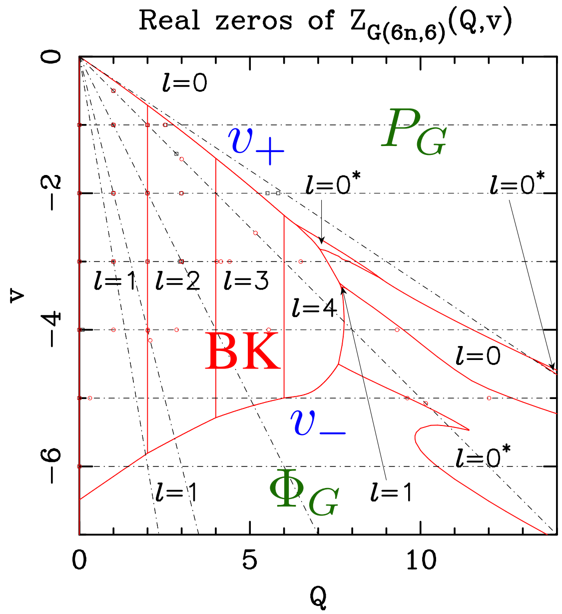
$$\beta_\ell = \sum_{\lambda \in S_\ell} \alpha_{\ell,\lambda} \dim \lambda\,, \qquad \gamma_{k+1} = \beta_{k+1} + \sum_{\ell=1}^{k} \sum_{\lambda \in S_\ell} \alpha_{\ell,\lambda} \dim \mathsf{D}_{L,\ell,\lambda}\,.$$

- For $\Phi_{G(nk,k)}(Q)$, we have extended this proof to $k = 7$.
- All the eigenvalues $\{\mu_{L,\ell,\lambda,s}, \mu_{L,k,s}, v^{2k}\}$ are distinct!
- **Conjecture 1**: These results are also true for any $k \ge 1$.

Practical procedure

- Fix $L = k+1$. For each $0 \leq \ell \leq k+1$ and $\lambda \in S_\ell$ compute symbolically the blocks $\mathsf{T}_{L,\ell,\lambda}$ and extract their non-trivial part $\mathsf{T}^{(nt)}_{L,\ell,\lambda}$. Each entry is a polynomial in both Q and v. (MATHEMATICA,PERL.)
- Compute the polynomial amplitudes $\{\alpha_{\ell,\lambda}(Q), \beta_k(Q), \gamma_{k+1}(Q)\}$.
- For selected values of $n \geq 1$, compute symbolically the traces $\mathrm{tr}(\mathsf{T}^{(nt)}_{L,\ell,\lambda})^n$. They are polynomials in Q, v. (MATHEMATICA, PERL, C.)
- Compute the symbolic Tutte polynomial $Z_{G(nk,k)}(Q,v)$.
  - For $\Phi_{G(7n,7)}$, the naive approach involved a TM of dimension $B(16) \approx 1.05 \times 10^{10}$. Our approach involved 31 blocks of $\dim \leq 11816$. For $n = 17$, CPU $\approx$ 30yr at LPTENS!!!!
- Find, on the real (Q,v)-plane, the roots of $Z_{G(nk,k)}$ along some lines $Q + pv = 0$ and $v = p$ with $p \in \mathbb{Z}$. (MPSOLVE WITH 50 DIGITS.)

Real zeros of $Z_{G(6n,6)}(Q, v)$

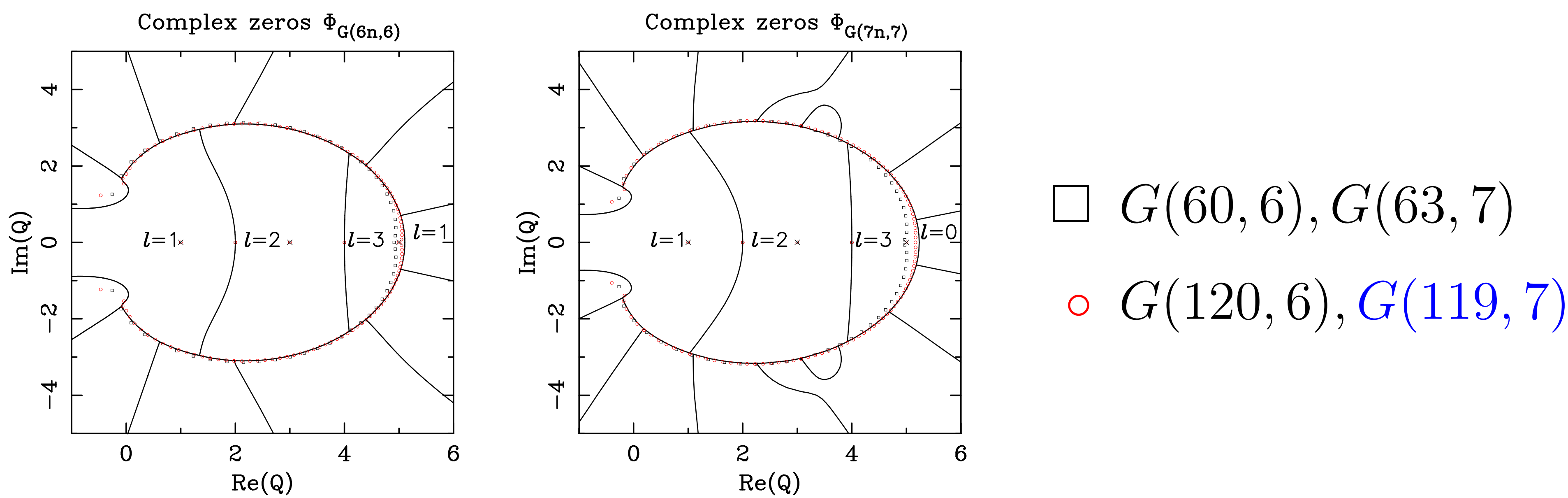


- "Phase diagram" of the Q-state Potts model on $G(6n, 6)$: the antiferromagnetic $v \in [-1, 0]$ and unphysical $v < -1$ regimes.
- Berker–Kadanoff phase: for each value of $0 \le Q \le Q_{\max}$, $v_-(Q) \le v \le v_+(Q)$.
- Zeros for $n = 3$ (□) and $n = 4$ (○).

- Zeros accumulate as $n \to \infty$ due to the Beraha–Kahane–Weiss Th.:
  (a) Limiting curves $\mathcal{B}$: There are two or more equimodular dominant eigenvalues at z_0. $|z_n - z_0| \sim 1/n$.
  - $Q = 2p$ $(p \ge 0)$ inside the BK phase, plus BK boundary.

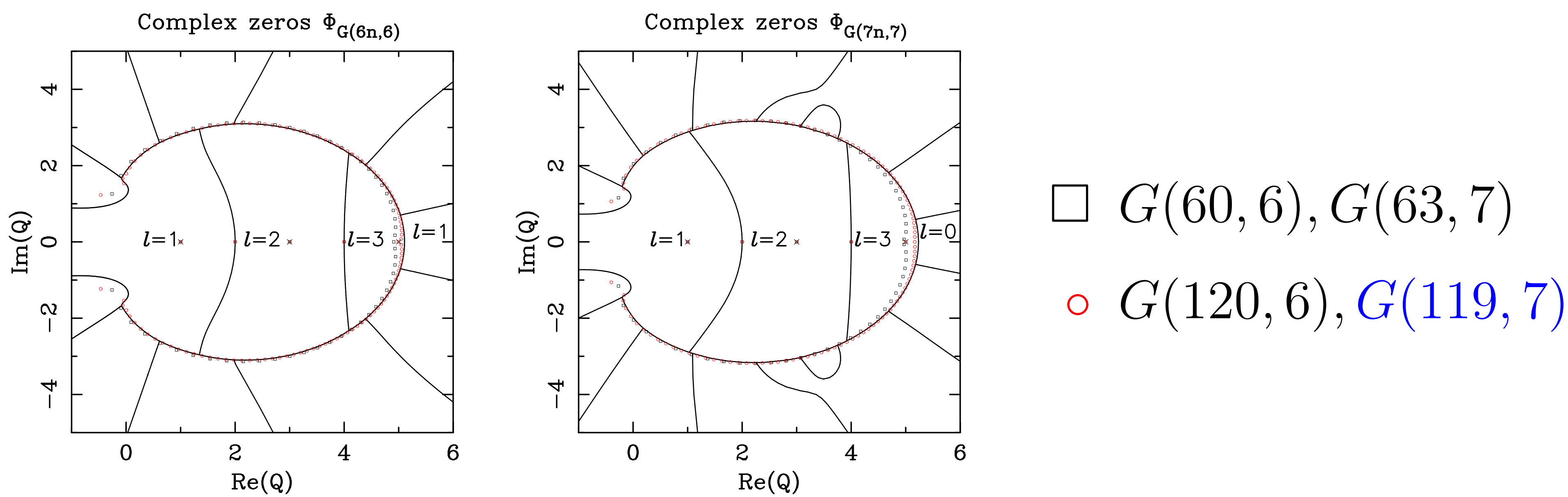
  (b) Isolated limiting points: There is a unique dominant eigenvalue μ_* at z_0, and $\alpha_*(z_0) = 0$. $|z_n - z_0| \sim r^n$ with $0 < r < 1$.
  - $Q = 2p + 1$ $(p \ge 0)$ inside the BK phase.

Real flow zeros of $\Phi_{G(kn,k)}(Q)$



- $\Phi_{G(119,7)}$ has two real roots $Q > 5$: $Q_1 \approx 5.0000197675$, and $Q_2 \approx 5.1653424423 \Rightarrow$ The HPR conjecture is FALSE!
- The familes $G(6n,6)$ and $G(7n,7)$ have an isolated limiting point at $Q = 5$. $\mu_* = \mu_{L,3,(3)}$ and $\alpha_{3,(3)}(Q) = \frac{1}{6}Q(Q-1)(Q-5)$.
- For $G(6n,6)$, there is a sequence of real flow roots converging to $Q = 5$ from below.

Real flow zeros of $\Phi_{G(kn,k)}(Q)$ (2)



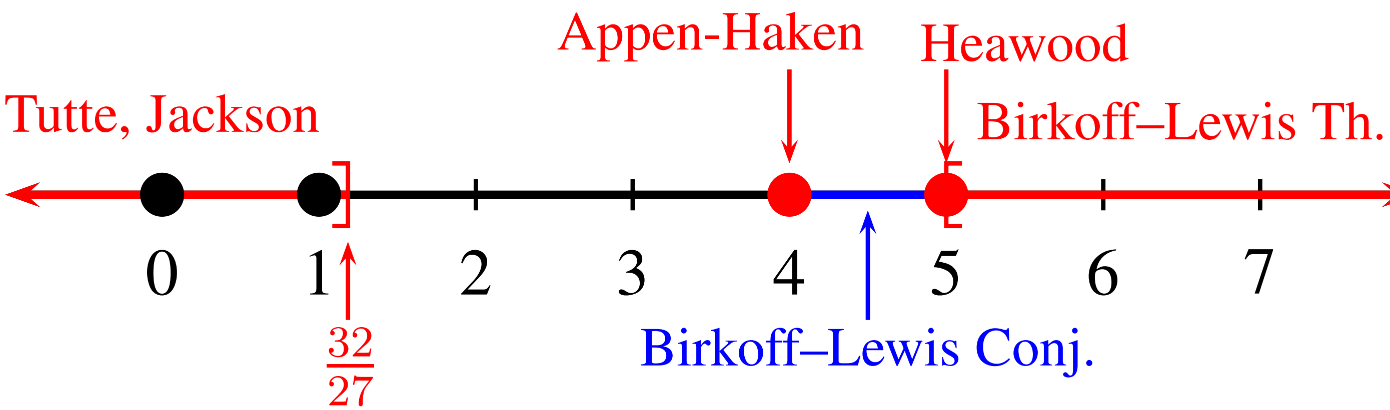
- For $G(7n,7)$, there are sub-sequences of real flow roots with odd n (resp. even n) converging to $Q = 5$ from above (resp. below).
- The proof/disproof of Tutte's 5-flow conjecture should be purely combinatorial (i.e., no complex-analysis involved!).
- Let $Q_c(k)$ be the largest real value of the limiting curve $\mathcal{B}_k$ for the family $G(nk,k)$.

Real flow zeros of $\Phi_{G(kn,k)}(Q)$ (3)

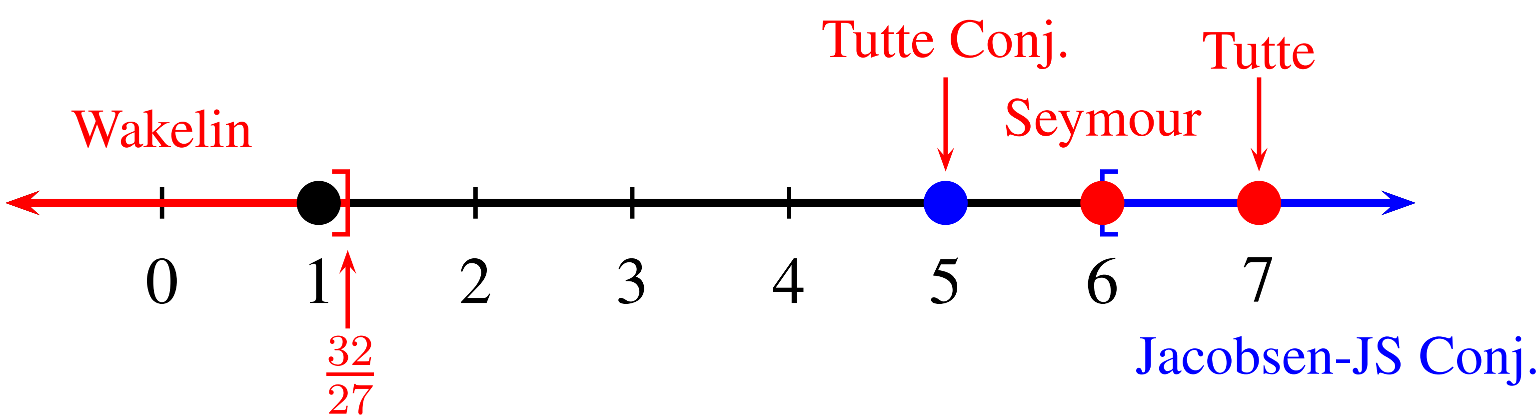
- For non-isolated limiting points Q_0, we can prove the existence of a sequence of real flow roots $Q_n \to Q_0$ by looking at the signs of the real analytic functions $\{\mu_{*,1}(Q), \alpha_{*,1}(Q)\}$ and $\{\mu_{*,2}(Q), \alpha_{*,2}(Q)\}$ on an appropriate real interval $[Q_0 - \epsilon, Q_0 + \epsilon]$.
- We can prove that $Q_c(7) \approx 5.2352605291$ is a non-isolated limiting point for the family $G(7n, 7)$, and there is a sequence of real flow roots converging to $Q_c(7)$ from below for odd n.
- We have computed the values of $Q_c(k)$ for $1 \leq k \leq 11$ and fitted to the Ansatz $Q_c(k) = Q_c(\infty) + Ak^{-\Delta}$. We obtained $Q_c(\infty) \approx 5.69$.
- **Conjecture 2** ($Q \to Q + 2$ scenario): For any bridgeless graph G, $\Phi_G(Q) > 0$ for any $Q \geq 6$.
- **Conjecture 3**: There does not exist any positive real number Q_0 such that for any bridgeless graph G, $\Phi_G(Q) > 0$ for any $Q \geq Q_0$.

Real zero-free intervals for P_G and Φ_G (3)

- P_G for planar graphs G: $P_G(Q) > 0$ for $Q \geq Q_0$ with $4 \leq Q_0 \leq 5$:

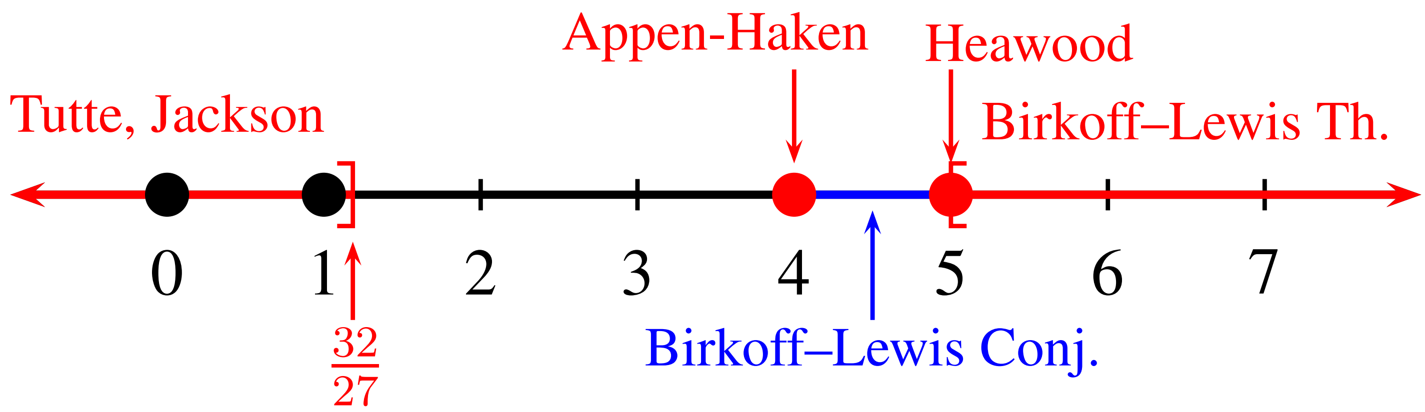


- Φ_G for bridgeless graphs G: $Q \to Q + 2$ scenario

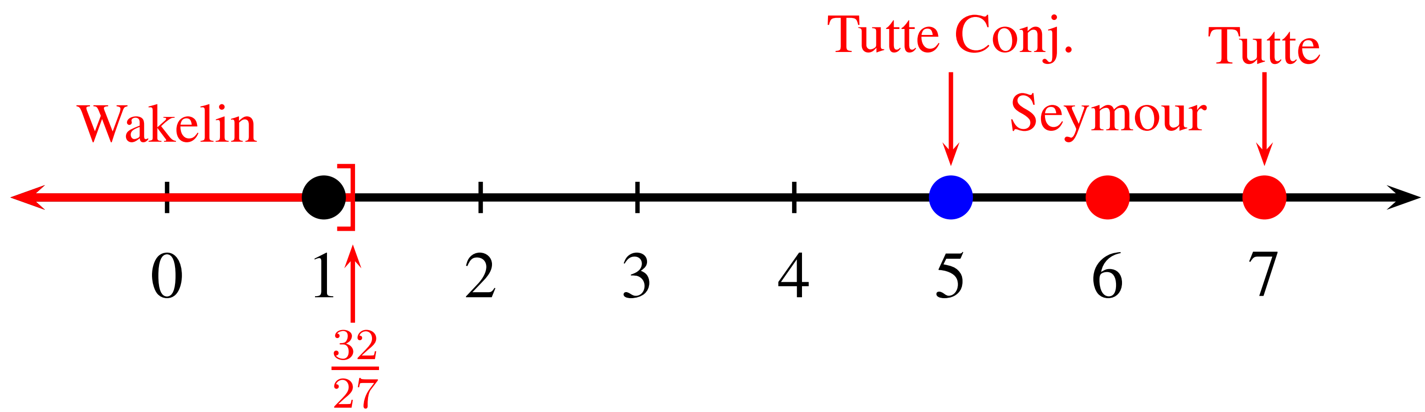


Real zero-free intervals for P_G and Φ_G (4)

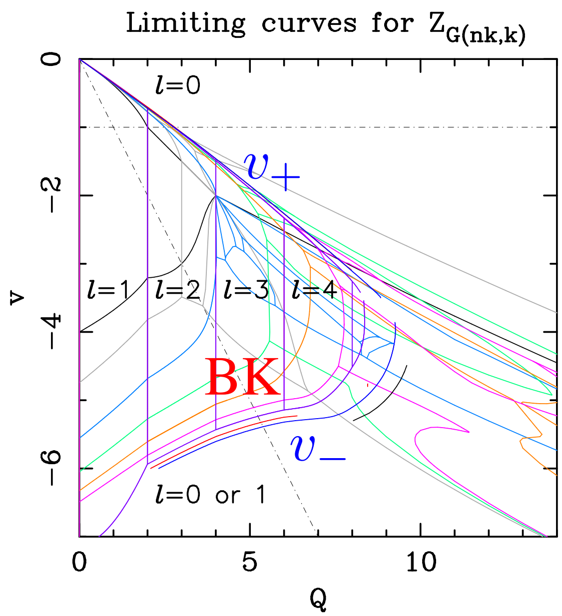
- P_G for planar graphs G: $P_G(Q) > 0$ for $Q \geq Q_0$ with $4 \leq Q_0 \leq 5$:



- Φ_G for bridgeless graphs G: No-uniform-upper-bound scenario



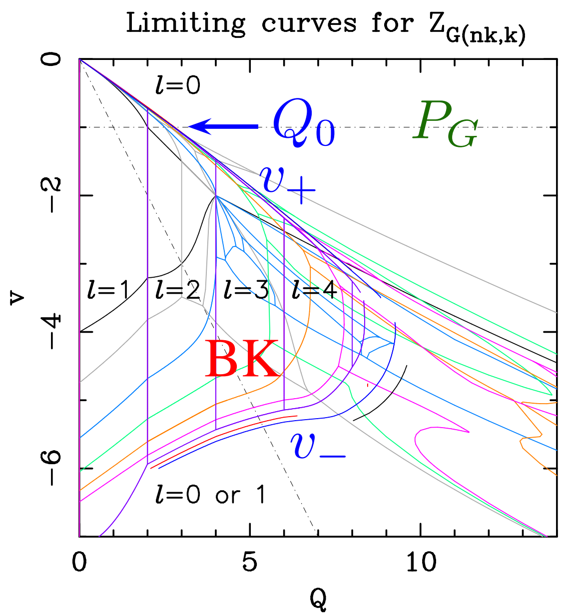
Berker–Kadanoff phase for the Petersen graphs



- We show the limiting curves $\mathcal{B}_k$ for the families $G(nk, k)$ with $1 \le k \le 11$.
- There is a "regular part" bounded by curves $v_{\pm}(Q)$ with vertical lines at $Q = 2p \le 8$.
- For large Q, there is also a non-regular part.

- For $1 \le k \le 6$, we find that for $p \ge 0$,
  - In the BK phase, the dominant eigenvalue in the interval $(2p, 2p+2)$ is $\mu_{L,p+1,(p+1)}$.
  - The corresponding amplitudes $\alpha_{p+1,(p+1)}(Q)$ vanish at $Q = 2p+1 \Rightarrow Q = 2p+1$ is an isolated limiting point
- **Conjecture 4**: These properties are also true for any $k \ge 1$, at least up to $Q \le Q_{\max}$ with $Q_{\max} \gtrsim 12.4(1)$.

Beraha conjecture for recursive planar graphs

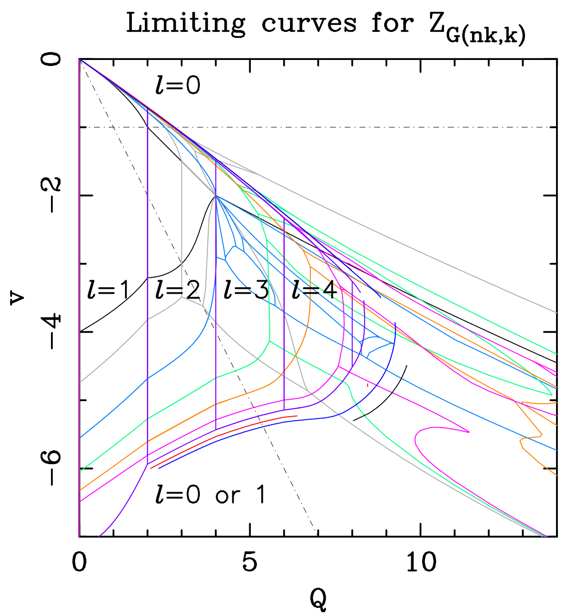


- We look for real accumulation points of chromatic zeros for a recursive family G_n.
- The *window* $0 \le Q \le Q_0(L)$ given by P_{G_n} is usually too small!

- On planar strip graphs [Saleur 1990–91]:
  - Zeros of Z_{G_n} accumulate around $Q = B_p$ in the whole BK phase.
  - At $Q = B_p$ there are massive cancellations of eigenvalues, and some amplitudes α_j vanish.
  - At $Q = B_p$, the physics is distinct from that of the BK phase!

$$\lim_{n\to\infty} \lim_{Q\to B_p} Z_{G_n}(Q,v) \neq \lim_{Q\to B_p} \lim_{n\to\infty} Z_{G_n}(Q,v) \,.$$

Beraha conjecture for the Petersen graphs



- The role played by $Q = B_p$ for planar graphs is played by the non-negative integers for $G(nk, k)$:
  - $Q = 2p$ are non-isolated limiting points.
  - $Q = 2p + 1$ are isolated limiting points.

- At $Q = n \geq 0$ there are massive eigenvalue cancellations, and some amplitudes vanish. (Inclusion-exclusion patterns.)

- The results on $G(nk, k)$ can be extended to other non-planar strip graphs, as the former are "almost" generically non-planar graphs.

- **Conjecture 5**: For any family $G_{L,n}$ of non-planar recursive graphs with n layers and cyclic boundary conditions, the same scenario found for the $G(nk, k)$ holds.

Conclusions

- Real flow zeros for bridgeless graphs:
  - The conjecture of Haggard–Pierce–Royle is false.
  - Perhaps there is a uniform upper bound $Q_{\max}$ such that for any bridgeless graph G, $\Phi_G(Q) > 0$ for any $Q \geq Q_{\max}$ (e.g., $Q_{\max} = 6$), or there is no such $Q_{\max}$.
  - $Q = 5$ is a limiting point of sequences of real flow zeros for some families of generalized Petersen graphs $G(nk, k)$.
- Beraha conjecture for non-planar strip graphs:
  - The role played by the Beraha numbers for planar graphs is played by the non-negative integers for non-planar graphs.
  - At $Q = n \in \mathbb{N} \cup \{0\}$ there are massive eigenvalue cancellations and amplitude vanishing.
  - The BK phase for the Q-state Potts model on non-planar strip graphs exists.

That's all folks!

Thanks a lot for your attention!!!