The Merino-Welsh Conjecture for Series-Parallel Graphs

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THE CONJECTURE(S) SERIES-PARALLEL GRAPHS

AUSTRALIA

Australia and Europe area comparison

Australia's area: 7.7 million sq km
Europe's area (shown): 3.5 million sq km

Darwin to Perth 4396 km - Perth to Adelaide 2707 km - Adelaide to Melbourne 726 km
Melbourne to Sydney 887 km - Sydney to Brisbane 972 km - Brisbane to Cairns 1748 km
THE CONJECTURE(S)  SERIES-PARALLEL GRAPHS

THE PARTICIPANTS I

Criel Merino

Dominic Welsh
The participants II

Activities & Hobbies

- The Tutte Polynomial
- Head of Department\textsuperscript{a} at Brunel
- CEROC
- 1930s jazz
- Wolverhampton Wanderers
- Amusing impersonations of famous mathematicians, especially after a beer or two.

\textsuperscript{a}Subject to change
Let \( G \) be a graph, and denote by:

- \( \tau(G) \), the number of \textit{spanning trees} of \( G \),
- \( \alpha(G) \), the number of \textit{acyclic orientations} of \( G \),
- \( \alpha^*(G) \), the number of \textit{totally cyclic orientations} of \( G \).

One version of the \textit{Merino-Welsh conjecture} asserts that if \( G \) is a loopless, bridgeless graph, then

\[
\tau(G)^2 \leq \alpha(G) \cdot \alpha^*(G).
\]
An orientation of a graph is the assignment of a direction to each edge:
Each orientation of $G$ determines a directed graph $D$.

- If $D$ has no directed cycles, then the orientation is *acyclic*.
- If every arc of $D$ lies in at least one directed cycle, then the orientation is *totally cyclic*.

For the Petersen graph, $\alpha^*(P) = 1920$, $\tau(P) = 2000$, and $\alpha(P) = 16680$. 
As is well known,

\[ \tau(G) = T_G(1, 1), \]
\[ \alpha(G) = T_G(2, 0), \]
\[ \alpha^*(G) = T_G(0, 2), \]

where \( T_G(x, y) \) is the Tutte Polynomial.

Thus we can extend from graphs to matroids in the usual way:\(^1\):

**Conjecture** For any loopless, coloopless matroid \( M \),

\[ T_M(1, 1)^2 \leq T_M(2, 0) \cdot T_M(0, 2). \]

\(^1\) i.e replacing \( G \) by \( M \)
The graph has two spanning trees,

\[\text{two}\] acyclic orientations

and \text{two} totally cyclic orientations:

\[\text{and so}\]

\[\tau = \alpha = \alpha^* = 2.\]
THE TUTTE PLANE

\[ \alpha^*(M) \]
\[ \tau(M) \]
\[ \alpha(M) \]
For any matroid $M$, the function

$$\varphi_M(x) = T_M(1 + x, 1 - x),$$

is \textit{convex} in the region $-1 \leq x \leq 1$. 

[Graph showing the convex function]
Chávez-Lomelí, Merino, Noble and Ramírez-Ibáñez proved this for all *paving matroids* (matroids with no circuits of size less than rank), and for all lines $x + y = p$ in the first quadrant.

Paving matroids form a huge class of matroids — probably containing *almost all matroids*.

But almost no graphs are paving matroids.
If a graph is *sparse* then it has lots of *acyclic* orientations:

**Theorem (Thomassen)** If $G$ is a simple graph on $n$ vertices with $m \leq \frac{16n}{15}$ edges, then $\tau(G) < \alpha(G)$.

If a graph is *dense* then it has lots of *totally cyclic* orientations:

**Theorem (Thomassen)** If $G$ is a bridgeless graph on $n$ vertices with $m \geq 4n - 4$ edges, then $\tau(G) < \alpha^*(G)$. 

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**GORDON ROYLE & STEVE NOBLE**

**THE MERINO-WELSH CONJECTURE**
**The Conjecture(s)  Series-Parallel Graphs**

**A possible approach**

**Question**

Thomassen asked whether it could possibly be the case that if $G$ is a graph with $n$ vertices and $m$ edges, that:

- $\tau(G) \leq \alpha(G)$ if $m \leq 2n - 2$, and
- $\tau(G) \leq \alpha^*(G)$ if $m \geq 2n - 2$.

If $m(G) \leq 2(n(G) - 1)$, then $m(G^*) \geq 2(n(G^*) - 1)$ so these are equivalent.
Suppose $G$ has $n - 2$ digons and 2 edges in a cycle; then

$$m(G) = 2n(G) - 2.$$

$$\tau(G) = 2^{n-1} + (n - 2)2^{n-3}$$
$$\alpha(G) = 2^n - 2$$
$$\alpha^*(G) = 2 \cdot 3^{n-2}$$

If $n \geq 6$ then

$$\alpha(G) < \tau(G) < \alpha^*(G).$$
A series-parallel graph\(^2\) is either:

- \(K_2\) with one vertex the \textit{sink} and the other the \textit{source}

- The \textit{series-connection} \(G_1 \oplus_S G_2\) of two smaller s/p graphs

- The \textit{parallel-connection} \(G_1 \oplus_P G_2\) of two smaller s/p graphs

\(^2\text{strictly, a 2-terminal series parallel graph}\)
**The Conjecture(s) Series-Parallel Graphs**

**Main Result**

**Theorem** If $G$ is a loopless bridgeless series-parallel graph, then

$$\tau(G)^2 \leq \alpha(G)\alpha^*(G).$$

The class of series-parallel graphs is minuscule, but

- They straddle the “critical point” where $m = 2n - 2$, and
- The proof does *not require* showing that $\alpha$ or $\alpha^*$ dominates.
We identify a recursion that allows us to compute $\tau$, $\alpha$, $\alpha^*$ for series and parallel connections.

We define a concept of *reducibility* such that a reducible graph cannot "be in" a minimal counterexample.

We devise a systematic procedure to find a minimal counterexample.

We execute this procedure and see what happens.
Counting spanning trees

- It is clear that $\tau(G \oplus_S H) = \tau(G) \cdot \tau(H)$
- But (in general) $\tau(G \oplus_P H)$ is not determined by $\tau(G)$, $\tau(H)$.

- If $\tau_2(G)$ denotes the number of 2-forests of $G$, then $\tau(G \oplus_P H)$ is determined by $\tau(G)$, $\tau(H)$, $\tau_2(G)$ and $\tau_2(H)$. 
A \textit{2-forest} of \((G, s, t)\) is a set of edges that induces a 2-component forest with \(s\) and \(t\) in different components.

A \textit{very acyclic} orientation of \((G, s, t)\) is an acyclic orientation with no directed path between \(s\) and \(t\).

An \textit{almost totally cyclic} orientation of \((G, s, t)\) is an orientation in which every edge is either in a directed cycle or a directed path between \(s\) and \(t\).

Use \(\tau_2(G)\), \(\alpha_2(G)\) and \(\alpha^*_2(G)\) to denote their respective numbers.
A graph\(^3\) \(G\) is \textit{reducible} if there is a \textit{smaller} graph \(H\), such that for \textit{all graphs} \(K\):

\[
\text{if } H \oplus_p K \text{ is ok , then } G \oplus_p K \text{ is ok,}
\]

where “ok” is shorthand for

“satisfies the Merino-Welsh conjecture.”

A minimal counterexample cannot “use” \(G\), because we could replace it by \(H\) instead.

\(^3\)henceforth, graph = 2-terminal series-parallel graph
How on earth can we test reducibility?

We want to show that $G$ is reducible, as witnessed by $H$.

$$\tau(H \oplus_P K) = \tau(H)\tau_2(K) + \tau_2(H)\tau(K)$$
How on earth can we *test* reducibility?

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\tau(H \oplus_P K) = \tau(H)\tau_2(K) + \tau_2(H)\tau(K)
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\]

So the effect of replacing $H$ by $G$ in *any* parallel connection can to some extent be quantified.
Claim: \( G \) is \textit{reducible}, with \( H \) witnessing this.
If $H \oplus_p K$ is ok, then for $t_1, t_2, a_1, a_2, c_1, c_2$ determined by $K$

$$(4t_1 + 4t_2)^2 \leq (4a_1 + 2a_2) \cdot (14c_1 + 4c_2).$$

Replacing $H$ with $G$, we have

$$(4t_1 + 4t_2)^2 \rightarrow (20t_1 + 25t_2)^2$$
$$(4a_1 + 2a_2) \rightarrow (62a_1 + 36a_2)$$
$$(14c_1 + 4c_2) \rightarrow (36c_1 + 18c_2)$$

Thus the left-hand side is multiplied by \textit{at most} $(25/4)^2$ and the right-hand side by \textit{at least} $(62/4) \cdot (36/14)$, and so $G \oplus_p K$ is ok.
We *systematically* make a list, starting with $K_2$ and then:

- Take (smallest) $G$ and $H$ from the list,
- Form $G \oplus_S H$ and $G \oplus_P H$,
- Test for reducibility (using smaller graphs on list) and add to list only if irreducible,
- Discard graphs with 3 (or more) edges in parallel or series.
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Every graph not on this list is *reducible*. 
AND THEN . . .

"I think you should be more explicit here in Step Two."
... the process terminates!
Let $G$ be a minimal counterexample

- $G$ is 2-connected
- $G = G_1 \oplus_P G_2$
- $G_1$ and $G_2$ are irreducible
- $G_1$ and $G_2$ are on the list

But, the list only contains a small number of tiny graphs, all of which satisfy the conjecture.
Conjecture true for series-parallel graphs,
Proof has no chance of generalising already been generalised!
New ideas needed, especially for $m = 2n - 2$,
Are there graphs with $m < 2n - 2$ edges with $\tau(G) > \alpha(G)$?