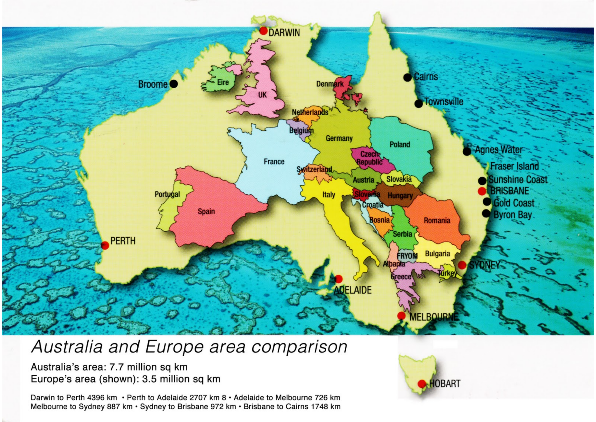
MERINO-WELSH CONJECTURE FOR SERIES-PARALLEL GRAPHS

Gordon Royle & Steve Noble

Centre for the Mathematics of Symmetry & Computation
School of Mathematics & Statistics
University of Western Australia
and
Mathematical Sciences
Brunel University

July 2015

AUSTRALIA


DARWIN
Broome
Eire
UK
Denmark
Cairns
Townsville
Netherlands
Belgium
Germany
Poland
France
Czech Republic
Agnes Water
Fraser Island
Switzerland
Austria
Slovakia
Sunshine Coast
BRISBANE
Portugal
Italy
Slovenia
Hungary
Gold Coast
Croatia
Byron Bay
Spain
Bosnia
Romania
Serbia
PERTH
FRYOM
Bulgaria
Albania
SYDNEY
Turkey
Greece
ADELAIDE
MELBOURNE
Australia and Europe area comparison
Australia's area: 7.7 million sq km
Europe's area (shown): 3.5 million sq km
Darwin to Perth 4396 km • Perth to Adelaide 2707 km 8 • Adelaide to Melbourne 726 km
Melbourne to Sydney 887 km • Sydney to Brisbane 972 km • Brisbane to Cairns 1748 km
HOBART

THE PARTICIPANTS I

Criel Merino

Dominic Welsh

The participants II

Steve Noble

Activities & Hobbies

- The Tutte Polynomial
- Head of Department[a] at Brunel
- CEROC
- 1930s jazz
- Wolverhampton Wanderers
- Amusing impersonations of famous mathematicians, especially after a beer or two.

[a] Subject to change

THE CONJECTURE(S)

Let G be a graph, and denote by:

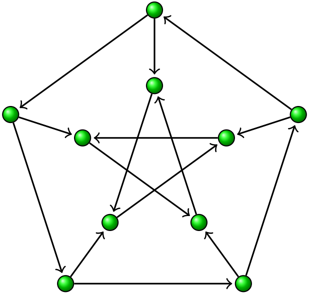
- $\tau(G)$, the number of *spanning trees* of G,
- $\alpha(G)$, the number of *acyclic orientations* of G,
- $\alpha^*(G)$, the number of *totally cyclic orientations* of G.

One version of the *Merino-Welsh conjecture* asserts that if G is a loopless, bridgeless graph, then

$$\tau(G)^2 \leq \alpha(G) \cdot \alpha^*(G).$$

ORIENTATIONS

An *orientation* of a graph is the assignment of a *direction* to each edge:



What about acyclic and totally cyclic?

Each orientation of G determines a directed graph D.

- If D has *no* directed cycles, then the orientation is *acyclic*.
- If every arc of D lies in *at least one* directed cycle, then the orientation is *totally cyclic*.

For the Petersen graph, $\alpha^*(P) = 1920$, $\tau(P) = 2000$, and $\alpha(P) = 16680.$

Duality

As is well known,

$$\tau(G) = T_G(1,1),$$
$$\alpha(G) = T_G(2,0),$$
$$\alpha^*(G) = T_G(0,2),$$

where $T_G(x,y)$ is the Tutte Polynomial.

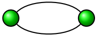
Thus we can extend from *graphs* to *matroids* in the usual way[1]:

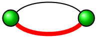
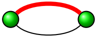
Conjecture For any loopless, coloopless matroid M,

$$T_M(1,1)^2 \leq T_M(2,0) \cdot T_M(0,2).$$

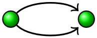
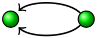
[1] i.e replacing G by M

EQUALITY

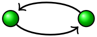
The graph  has *two* spanning trees,

two acyclic orientations

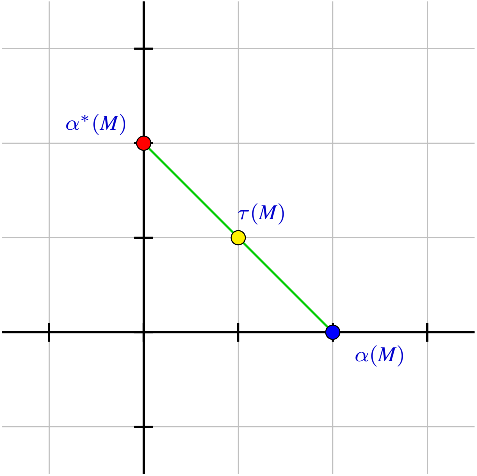
and *two* totally cyclic orientations:

and so

$$\tau = \alpha = \alpha^* = 2.$$

THE TUTTE PLANE

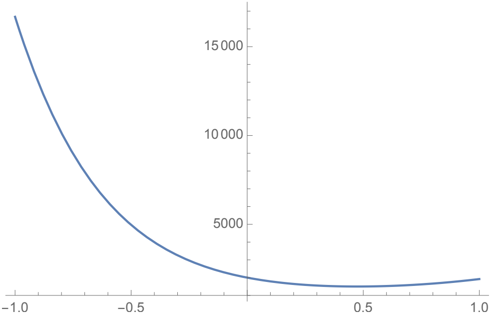


CONTINUOUS VERSION

For any matroid M, the function

$$\varphi_M(x) = T_M(1+x, 1-x),$$

is *convex* in the region $-1 \leq x \leq 1$.



PAVING MATROIDS

Chávez-Lomelí, Merino, Noble and Ramírez-Ibáñez proved this for all *paving matroids* (matroids with no circuits of size less than rank), and for all lines $x + y = p$ in the first quadrant.

Paving matroids form a huge class of matroids — probably containing *almost all matroids*.

But almost no graphs are paving matroids.

SPARSE GRAPHS

If a graph is *sparse* then it has lots of *acyclic* orientations:

THEOREM *(Thomassen) If G is a* simple *graph on n vertices with $m \leq 16n/15$ edges, then $\tau(G) < \alpha(G)$.*

If a graph is *dense* then it has lots of *totally cyclic* orientations:

THEOREM *(Thomassen) If G is a* bridgeless *graph on n vertices with $m \geq 4n - 4$ edges, then $\tau(G) < \alpha^*(G)$.*

A POSSIBLE APPROACH

QUESTION

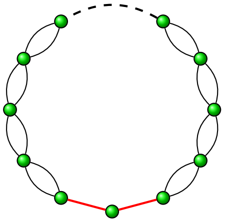
Thomassen asked whether it could possibly be the case that if G is a graph with n vertices and m edges, that:

- $\tau(G) \leq \alpha(G)$ if $m \leq 2n-2$, and
- $\tau(G) \leq \alpha^*(G)$ if $m \geq 2n-2$.

If $m(G) \leq 2(n(G)-1)$, then $m(G^*) \geq 2(n(G^*)-1)$ so these are equivalent.

But they're not true

Suppose G has $n-2$ digons and 2 edges in a cycle; then $m(G) = 2n(G) - 2$.



$$\tau(G) = 2^{n-1} + (n-2)2^{n-3}$$
$$\alpha(G) = 2^n - 2$$
$$\alpha^*(G) = 2 \cdot 3^{n-2}$$

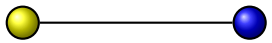
If $n \geq 6$ then

$$\alpha(G) < \tau(G) < \alpha^*(G).$$

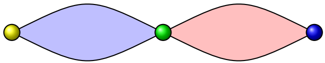
SERIES / PARALLEL GRAPHS

A *series-parallel graph*[2] is either:

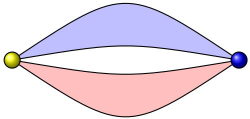
- K_2 with one vertex the *sink* and the other the *source*



- The *series-connection* $G_1 \oplus_S G_2$ of two smaller s/p graphs



- The *parallel-connection* $G_1 \oplus_P G_2$ of two smaller s/p graphs



[2] strictly, a 2-terminal series parallel graph

Main Result

Theorem If G is a loopless bridgeless series-parallel graph, then

$$\tau(G)^2 \leq \alpha(G)\alpha^*(G).$$

The class of series-parallel graphs is minuscule, but

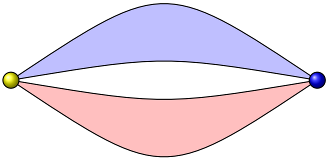
- They straddle the "*critical point*" where $m = 2n - 2$, and
- The proof does *not require* showing that α or α^* dominates.

Proof Flavour

- We identify a recursion that allows us to compute τ, α, α^* for series and parallel connections
- We define a concept of *reducibility* such that a reducible graph cannot *"be in"* a minimal counterexample
- We devise a systematic procedure to find a minimal counterexample
- We execute this procedure and see what happens

Counting spanning trees

- It is clear that $\tau(G \oplus_S H) = \tau(G) \cdot \tau(H)$
- But (in general) $\tau(G \oplus_P H)$ is not determined by $\tau(G)$, $\tau(H)$.



- If $\tau_2(G)$ denotes the number of *2-forests* of G, then $\tau(G \oplus_P H)$ *is determined* by $\tau(G)$, $\tau(H)$, $\tau_2(G)$ and $\tau_2(H)$.

KEEPING COUNT

- A 2*-forest* of (G, s, t) is a set of edges that induces a 2-component forest with s and t in different components.
- A *very acyclic* orientation of (G, s, t) is an acyclic orientation with no directed path between s and t.
- An *almost totally cyclic* orientation of (G, s, t) is an orientation in which every edge is either in a directed cycle or a directed path between s and t.

Use $\tau_2(G)$, $\alpha_2(G)$ and $\alpha_2^*(G)$ to denote their respective numbers.

REDUCIBILITY

A graph[3] G is *reducible* if there is a *smaller* graph H, such that for *all graphs* K:

if $H \oplus_P K$ is ok , then $G \oplus_P K$ is ok,

where "ok" is shorthand for

"satisfies the Merino-Welsh conjecture."

A minimal counterexample cannot "use" G, because we could replace it by H instead.

[3] henceforth, graph = 2-terminal series-parallel graph

Testing reducibility

How on earth can we *test* reducibility?

We want to show that G is reducible, as witnessed by H.

$$\tau(H \oplus_P K) = \tau(H)\tau_2(K) + \tau_2(H)\tau(K)$$

Testing reducibility

How on earth can we *test* reducibility?

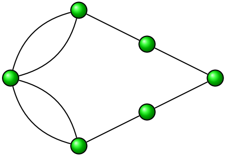
We want to show that G is reducible, as witnessed by H.

$$\tau(H \oplus_P K) = \tau(H)\tau_2(K) + \tau_2(H)\tau(K)$$
$$\tau(G \oplus_P K) = \tau(G)\tau_2(K) + \tau_2(G)\tau(K)$$

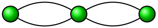
So the effect of replacing H by G in *any* parallel connection can to some extent be quantified.

REDUCIBILITY

G



H



$$(\tau(G), \tau_2(G), \alpha(G), \alpha_2(G), \alpha_2^*(G), \alpha_2(G)) = (20, 25, 62, 36, 36, 18)$$
$$(\tau(H), \tau_2(H), \alpha(H), \alpha_2(H), \alpha_2^*(H), \alpha_2(H)) = (4, 4, 4, 2, 14, 4)$$

Claim: G is *reducible*, with H witnessing this.

Reducibility

If $H \oplus_P K$ is ok, then for t_1, t_2, a_1, a_2, c_1, c_2 determined by K

$$(4t_1 + 4t_2)^2 \leq (4a_1 + 2a_2) \cdot (14c_1 + 4c_2).$$

Replacing H with G, we have

$$(4t_1 + 4t_2)^2 \to (20t_1 + 25t_2)^2$$
$$(4a_1 + 2a_2) \to (62a_1 + 36a_2)$$
$$(14c_1 + 4c_2) \to (36c_1 + 18c_2)$$

Thus the left-hand side is multiplied by *at most* $(25/4)^2$ and the right-hand side by *at least* $(62/4) \cdot (36/14)$, and so $G \oplus_P K$ is ok.

An algorithm

We *systematically* make a list, starting with K_2 and then:

- Take (smallest) G and H from the list,
- Form $G \oplus_S H$ and $G \oplus_P H$,
- Test for reducibility (using smaller graphs on list) and add to list only if irreducible,
- Discard graphs with 3 (or more) edges in parallel or series.

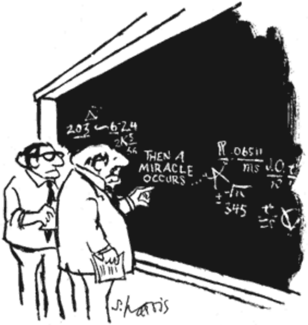
AN ALGORITHM

We *systematically* make a list, starting with K_2 and then:

- Take (smallest) G and H from the list,
- Form $G \oplus_S H$ and $G \oplus_P H$,
- Test for reducibility (using smaller graphs on list) and add to list only if irreducible,
- Discard graphs with 3 (or more) edges in parallel or series.

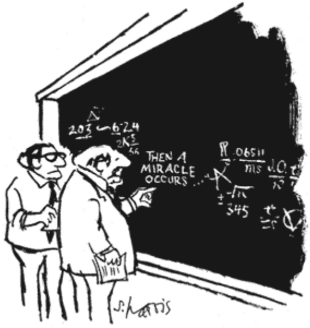
Every graph not on this list is *reducible*.

And then . . .



"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."

And then . . .



"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."

. . . the process terminates!

THE DÉNOUEMENT

Let G be a minimal counterexample

- G is 2-connected
- $G = G_1 \oplus_P G_2$
- G_1 and G_2 are irreducible
- G_1 and G_2 are on the list

But, the list only contains a small number of tiny graphs, all of which satisfy the conjecture.

CONCLUSION

- Conjecture true for series-parallel graphs,
- Proof has ~~*no chance of generalising*~~ already been generalised!
- New ideas needed, especially for $m = 2n - 2$,
- Are there graphs with $m < 2n - 2$ edges with $\tau(G) > \alpha(G)$?