Euler Characteristics and Chromatic Polynomials

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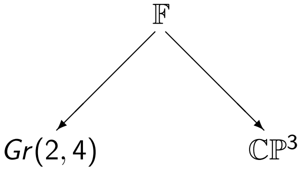
13th July 2015

Outline

1. In a wild but brief digression we indicate how the double fibration given in Alex Fink's talk is used in twistor theory.
2. We review some joint work with Mike Eastwood in which, given a graph, one constructs a family of manifolds whose Euler characteristics are the values of the chromatic polynomial of the graph at various integers.
3. We consider recent attempts to improve this result and extend it to the Tutte polynomial.
4. Acknowledgements and references.



$Gr(2,4)$ is complexified compactified Minkowski space $\mathcal{M}$, and in this context the $\mathbb{CP}^3$ is *twistor space*. For certain open subsets U of $\mathcal{M}$ and the corresponding subsets V of $\mathbb{CP}^3$ the *Penrose transform* gives an isomorphism between $H^1(V; \mathcal{O}(n))$ and Dirac fields, such as the electromagnetic field, on U.

Let Y be a closed submanifold (of codimension m) of the manifold X, and suppose that Y has orientable normal bundle. Then the Leray sequence is

$$\cdots \to H_i(X-Y) \stackrel{i}{\to} H_i(X) \stackrel{\cap}{\to} H_{i-m}(Y) \stackrel{\delta}{\to} H_{i-1}(X-Y) \to \dots .$$

The connecting homomorphism $\delta : H_{i-m}(Y) \to H_{i-1}(X-Y)$ replaces each point of a cycle by a small normal $m-1$ sphere.

Now choose Y to be the diagonal Δ in the space $X = M \times M$, where M is an m-dimensional orientable manifold. In $M \times M - \Delta$ the two variables have to avoid each other, in $M \times M$ they may coincide, and in Δ they must coincide.

Given a graph G with vertex set $V = \{v_1, \ldots, v_n\}$ and edge set E we construct a manifold M_G (of dimension mn) as follows. Let M be an m-dimensional manifold (to be specified later). In the product space $M^{\times n}$ each copy of M is associated to a particular vertex in G. Each edge $e = v_i v_j$ in G defines a "diagonal" subspace Δ_e of $M^{\times n}$ by

$$\Delta_e = \{(x_1, \ldots, x_n) \in M^{\times n} : x_i = x_j\}.$$

Now

$$M_G = M^{\times n} - \cup_{e \in E} \Delta_e.$$

We apply the Leray sequence to one edge, say f, at a time. Let

$$X - Y = M_G, \quad X = M^{\times n} - \cup_{e \in E, e \neq f} \Delta_e, \quad Y = X \cap \Delta_f,$$

so that $M_{G \setminus f} = X$ and $M_{G/f} = Y$. Then the complete long exact sequence is

$$0 \to H_{mn}(M_G) \to H_{mn}(M_{G \setminus f}) \to H_{mn-m}(M_{G/f}) \to H_{mn-1}(M_G) \to \cdots \to 0.$$

The alternating sum of the Betti numbers in this sequence is zero:

$$\beta_{mn}(M_G) - \beta_{mn}(M_{G\setminus f}) + \beta_{mn-m}(M_{G/f}) - \beta_{mn-1}(M_G) + \cdots = 0,$$

and hence, for even m as we now suppose, the Euler characteristics satisfy

$$\chi(M_G) - \chi(M_{G\setminus f}) + \chi(M_{G/f}) = 0.$$

Now we define our "atomic" space M in such a way that when G consists of just one vertex then $\chi(M)$ is the number of ways of colouring G with λ colours. We choose $M = \mathbb{CP}^{\lambda-1}$. Then $\chi(M) = \lambda$.

Theorem (Eastwood and Huggett)

Let G be a graph and M_G the generalised configuration space defined above, with $M = \mathbb{CP}^{\lambda-1}$. Then

$$\chi(G, \lambda) = \chi(M_G).$$

If G has a subgraph H such that those vertices in H which are joined to vertices in $G - H$ form a complete subgraph of H, then M_G is a fibration over M_H

$$\begin{array}{ccc} F & \hookrightarrow & M_G \\ & & \downarrow \\ & & M_H \end{array}$$

for some fibre F, and hence

$$\chi(M_G) = \chi(F)\chi(M_H).$$

Can we relax the conditions on M, that it is even-dimensional with orientable normal bundle?

Use Poincaré duality to switch to compactly supported cohomology:

$$\chi(M_G) = \Sigma_i (-1)^i \beta_c^i(M_G).$$

Let $\mathcal{S}_{M_G}$ be the "skyscraper sheaf" on $M^{\times n}$ with support on M_G, so that $H^i_c(M_G) = H^i_c(M^{\times n}; \mathcal{S}_{M_G})$. Given any edge f of G there is a short exact sequence of sheaves on $M^{\times n}$

$$0 \to \mathcal{S}_{M_G} \to \mathcal{S}_{M_{G \setminus f}} \to \mathcal{S}_{M_{G/f}} \to 0.$$

The corresponding long exact sequence in cohomology with compact supports is Poincaré dual to the Leray sequence, and arguing as before we find that

$$\chi(G, \lambda) = \Sigma_i (-1)^i \beta^i_c(M_G).$$

This allows us to choose a different "atomic" space M. One possibility is to let M be the Riemann sphere with $\lambda + 2$ points blown up, so that its Euler characteristic is $-\lambda$, as required.

But what about the Tutte polynomial? The most challenging question seems to be the geometrical description of multiple edges and loops. For this, we are exploring a different short exact sequence.

As before let X be a manifold, but this time choose Y to be a divisor (and hence of codimension 1). Denote by $[Y]$ the associated line bundle of Y, and denote by K_X the canonical bundle of X. Then

$$0 \to K_X \xrightarrow{i} K_X \otimes [Y] \xrightarrow{\rho} K_Y \to 0$$

is exact, where ρ is the Poincaré residue map.

As before, choose X and Y so that K_X corresponds to the graph $G \backslash f$, $K_X \otimes [Y]$ corresponds to G, and K_Y corresponds to G/f. Then the holomorphic Euler characteristic of our short exact sequence gives part of the recursion formula for the Tutte polynomial.

In the case of an edge with multiplicity m, deletion and contraction becomes

$$0 \to K_X \otimes [(m-1)Y] \to K_X \otimes [mY] \to (K_X \otimes [mY])|_Y \to 0$$

so now we have to identify $(K_X \otimes [mY])|_Y$.

For example, if $X = \mathbb{CP}^1 \times \mathbb{CP}^1$ and Y is the diagonal, then

$$(K_X \otimes [mY])|_Y = \mathcal{O}_{\mathbb{CP}^1}(-2 + 2(m-1)).$$

We will need to calculate other examples see whether they can plausibly be interpreted as corresponding to loops.

I am very grateful to the following people for their observations and suggestions:

Andrea D'Agnolo (Padua)
Gerd Dethloff (Brest)
Daniel Robertz (Plymouth)

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M Eastwood and S Huggett
European Journal of Combinatorics **28** (2007) 1553–1560

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