On Postnikov-Shapiro algebras

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Introduction

Few commutative algebras presented in this poster, for graphs, vector configurations and hypergraphs, whose k-graded components have combinatorial sense (Hilbert Series) and can be calculated through a Tutte-polynomial. Combinatorial sense is expressed in terms of spanning trees or subforests and their externally activity.

Let G some graph (in following boxes with n vertices). Fix some linear order on the edges of G. Let F be any subforest in G. By $act_G(F)$ denote the number of all externally active edges of F, i.e. the number of edges $e \in G \setminus F$ such that subgraph $F+e$ has a cycle and e is the minimal edge in this cycle in the above linear order.

Algebras for spanning trees

1st Let Φ_G^T be the algebra generated by $\{\phi_e : e \in G\}$ with relations $\phi_e^2 = 0$, for any $e \in G$, and $\prod_{e \in H} \phi_e = 0$, for any cut $H \subset G$.

Take any linear order of vertices of G. For $i = 1, \ldots, n$, set

$$X_i = \sum_{e \in G} c_{i,e}\phi_e,$$

where $c_{i,e} = \pm 1$ for vertices incident to e (for the smaller vertex, $c_{i,e} = 1$, for the bigger vertex, is $c_{i,e} = -1$) and 0 otherwise. Denote by $\mathcal{C}_G^T$ the subalgebra of Φ_G^T generated by $X_1, \ldots, X_n$.

2nd Let $\mathbb{K}$ be some field of characteristic 0. Consider the ideal J_G^T in the ring $\mathbb{K}[x_1, \cdots, x_n]$ generated by

$$p_I^T = \left(\sum_{i \in I} x_i\right)^{D_I},$$

where I ranges over all nonempty subsets of vertices, and D_I is the total number of edges from vertices in I to vertices outside the subset I. Define the algebra $\mathcal{B}_G^T$ as the quotient $\mathbb{K}[x_1, \ldots, x_n]/J_G^T$.

Theorem(Postnikov, Shapiro) *The algebras B_G^T and C_G^T are isomorphic. Their total dimension as vector spaces over $\mathbb{K}$ is equal to the number of spanning trees in the graph G.*

The dimension of the k-th graded component of these algebras equals the number of spanning trees T of G with external activity $|G| - n + 1 - k$.

Corollary *Dimension of the k-th graded component is equal to the coefficient of the monomial $y^{|G|-n+1-k}$ in the polynomial $T_G(1, y)$.*

Algebras and Matroids

Theorem *For graphs G_1 and G_2 algebras $B_{G_1}^F$ and $B_{G_2}^F$ are isomorphic if and only if graphical matroids corresponds to graphs are same.*

Corollary *For any graph G, integer a and b, we can construct algebra $B_G^{F_b}$ knowing only algebra $B_G^{F_a}$ and a, b.*

Corollary *For any graph G and integer t, we can compute Tutte-polynomial knowing only algebra $B_G^{F_t}$ and t.*

Remark There are same results for Algebras corresponds to spanning trees, however only for connected graphs.

Algebras for subforests

1st Let $\Phi_G^{F_t}$ be the algebra generated by $\{\phi_e : e \in G\}$ with relations $\phi_e^{t+1} = 0$, for any $e \in G$.

Take any linear order of vertices of G. For $i = 1, \ldots, n$, set

$$X_i = \sum_{e \in G} c_{i,e}\phi_e,$$

where $c_{i,e} = \pm 1$ for vertices incident to e (for the smaller vertex, $c_{i,e} = 1$, for the bigger vertex, is $c_{i,e} = -1$) and 0 otherwise. Denote by $\mathcal{C}_G^{F_t}$ the subalgebra of $\Phi_G^{F_t}$ generated by $X_1, \ldots, X_n$.

2nd Let $\mathbb{K}$ be some field of characteristic 0. Consider the ideal $J_G^{F_t}$ in the ring $\mathbb{K}[x_1, \cdots, x_n]$ generated by

$$p_I^F = \left(\sum_{i \in I} x_i\right)^{t \cdot D_I + 1},$$

where I ranges over all nonempty subsets of vertices, and D_I is the total number of edges from vertices in I to vertices outside the subset I. Define the algebra $\mathcal{B}_G^{F_t}$ as the quotient $\mathbb{K}[x_1, \ldots, x_n]/J_G^{F_t}$.

Remark For $t = 1$ these algebras (denoted by $\mathcal{B}_G^F$ and $\mathcal{C}_G^F$) were introduced by A. Postnikov and B. Shapiro, the following result was proved

Theorem(Postnikov, Shapiro) *The algebras B_G^F and C_G^F are isomorphic. Their total dimension as vector spaces over $\mathbb{K}$ is equal to the number of subforests in the graph G.*

The dimension of the k-th graded component of these algebras equals the number of subforests F of G with external activity $|G| - |F| - k$.

A subforest of the graph G with a label from 1 to t on each edge is called a t-labeled forest. The weight of a t-labeled forest F (denoted by $\omega(F)$) is the sum of labels of all its edges.

Theorem *For any graph G and a positive integer t, algebras $\mathcal{B}_G^{F_t}$ and $\mathcal{C}_G^{F_t}$ are isomorphic, their total dimension over $\mathbb{K}$ is equal to the number of t-labeled forests in G.*

The dimension of the k-th graded component of the algebra $\mathcal{B}_G^{F_t}$ is equal to the number of t-labeled forests F of G with the weight $t \cdot (e(G) - act_G(F)) - k$.

Corollary *Dimension of the k-th graded component of $\mathcal{B}_G^{F_t}$ is equal to the coefficient of the monomial $y^{t \cdot e(G) - c(G) + v(G) + 1 - k}$ in the polynomial*

$$\left(\frac{y^t - 1}{y - 1}\right)^{v(G) - c(G)} \cdot T_G\left(\frac{y^{t+1} - 1}{y^{t+1} - y}, y^t\right),$$

where $c(G)$ is number of connected components

Theorem *For any positive integer $t \geq n$, it is possible to restore the Tutte polynomial of any graph G on n vertices knowing only the dimensions of each graded component of the algebra $\mathcal{B}_G^{F_t}$.*

Vector Configurations

Given set A of vectors $a_1, \ldots, a_m$ in $\mathbb{K}^n$. Let Φ_m^F be the algebra generated by $\{\phi_i : i \in [1..m]\}$ with relations $\phi_i^2 = 0$, for any $i \in [1..m]$.

For $i = 1, \ldots, n$, set $X_i = \sum_{k \in [1..m]} a_{k,i}\phi_k$. Denote by $\mathcal{C}_A^F$ the subalgebra of Φ_m^F generated by $X_1, \ldots, X_n$.

It was introduced by A. Postnikov, B. Shapiro and M. Shapiro. The following theorem is trivial consequence of their results

Theorem *It is possible to calculate Hilbert Series of $\mathcal{C}_A^F$ knowing only Tutte polynomial of vector matroid corresponds to A.*

Proposition *The algebra is not changed by a change of basis of $\mathbb{K}^n$.*

Remark It is impossible to construct such algebra knowing only vector matroid of vector configuration A.

Theorem *It is possible to restore vector matroid from such Algebra.*

Hypergraphs

Given a hypergraph H on n vertices, let as associate commuting variables $\phi_e, e \in H$ to all edges of H.

Set Φ_H^F be the algebra generated by $\{\phi_e : e \in H\}$ with relations $\phi_e^2 = 0$, for any $e \in H$.

Let $C = c_{i,e} : i \in [1, n], e \in E$ is set of parametries, such that $c_{i,e} = 0$ for vertices noincident to e, and $\sum_{i=1}^n c_{i,e} = 0$.

For $i = 1, \ldots, n$, set

$$X_i = \sum_{e \in H} c_{i,e}\phi_e,$$

Denote by $\widehat{\mathcal{C}_H}(C)$ the subalgebra of Φ_H^F generated by $X_1, \ldots, X_n$, and by $\widehat{\mathcal{C}_H}$ all set of such subalgebras.

Proposition *For two proportional sets of parameters, corresponding algebras are isomorphic.*

Theorem *Almost all algebras from $\widehat{\mathcal{C}_H}$ have same Hilbert Series, and this HS is possible to calculate from Tutte polynomial of "general" matroid corresponds to H.*

Remark For usual graph G, almost all algebras from $\widehat{\mathcal{C}_G}$ are isomorphic to $\mathcal{C}_G$.