Theta Graphs, Bicliques and the Alpha Conjectures.

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Workshop on New Directions for the Tutte polynomial, 2015

Overview

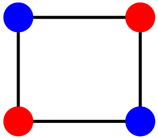
- Chromatic Polynomial
- Chromatic Roots
- Alpha Conjectures
- Cycles
- Bicliques
- Generalised Theta Graphs

The Chromatic Polynomial

The **Chromatic Polynomial** counts the number of proper colourings in at most λ colours.

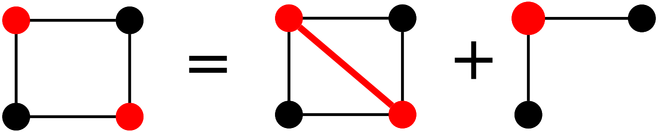
$$P(G;\lambda) = (-1)^{n-k(G)}\lambda^{k(G)}T(G;1-\lambda,0).$$

$$P(G;\lambda) = \lambda(\lambda-1)(\lambda^2-3\lambda+3)$$



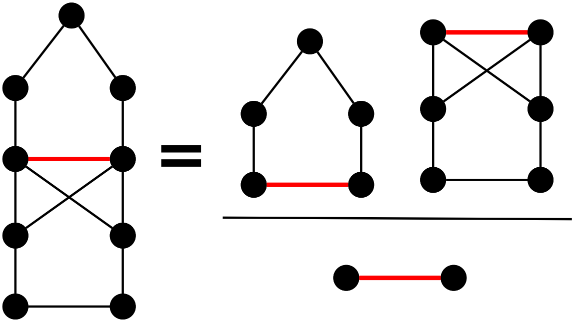
Addition-Identification

$$P(G;\lambda) = P(G+uv;\lambda) + P(G/uv;\lambda),\ uv \notin E(G).$$



Clique-separable Graphs

$$P(G;\lambda) = \frac{P(H_1;\lambda) \times P(H_2;\lambda)}{P(K_r;\lambda)}.$$



The Chromatic Polynomial - Properties

Cycles

$$\begin{aligned} P(C_n;\lambda) &= (\lambda-1)^n + (-1)^n(\lambda-1) \\ &= (-1)^{n-1}(\lambda-1)((\lambda-1)^{n-1}-1). \end{aligned}$$

Complete Graphs

$$P(K_r;\lambda) = \lambda(\lambda-1)\ldots(\lambda-r+1).$$

Motivation

Chromatic Roots

- Large amount of research on chromatic roots
  - Integer roots
  - Real roots for various classes of graphs
  - Complex roots
- Algebraic theory of chromatic roots is *"as yet rather undeveloped"* (Sokal, 2004)

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Chromatic Roots

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Motivation

Chromatic Roots

- "Which algebraic integers are chromatic roots?"
  - Is $B(10)$ a chromatic root?[a]
  - Can i be a chromatic root? (Dong, 2008)
  - $\alpha + n$ conjecture (Cameron, et al., 2008)
  - $n\alpha$ conjecture (Cameron, et al., 2008)
- Chromatic factorisation
- Chromatic equivalence
- Splitting field equivalence
- Galois groups

[a] Noninteger $B(n) = 2 + 2\cos\frac{2\pi}{n}$, $n \neq 10$, are not chromatic roots.

Chromatic Roots

Chromatic Roots

Chromatic Roots

- Integer roots: $0, 1, \ldots, \chi(G) - 1$.
- No real roots in $(-\infty, 0) \cup (0, 1) \cup \left(1, \frac{32}{27}\right]$ (Jackson, 1993).
- Complex roots are dense in complex plane (Sokal, 2003).
    - Complex roots of generalised theta graphs are dense in complex plane (possibly excluding disc $|\lambda - 1| < 1$).

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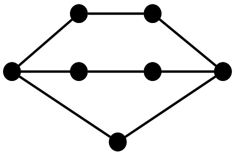
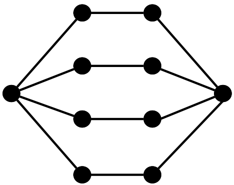
Generalised Theta Graphs

- $\Theta_{a_0,a_1,\ldots,a_k}$, $k \geq 1$
- $k+1$ paths
- Length of i-th path is a_i
- $\Theta_{a^{k+1}}$ has $k+1$ paths of length a
- Complex roots of $\Theta_{a^{k+1}}$ are dense in complex plane



$\Theta_{2,3,3}$



$\Theta_{3,3,3,3} = \Theta_{3^4}$

Which numbers can be chromatic roots?

Which numbers can be chromatic roots?

The chromatic polynomial is a **monic** polynomial with **integer** coefficients.

Chromatic roots are **algebraic integers**.

Not all algebraic integers are chromatic roots.

Question

Which algebraic integers are chromatic roots?

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Question

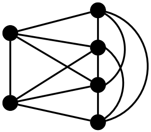
Which algebraic integers are chromatic roots?

Algebraic Integers

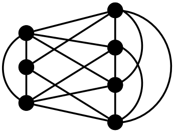
Conjecture ($\alpha + n$ Conjecture (Cameron))

For any algebraic integer α there is an $n \in \mathbb{N}$ such that $\alpha + n$ is a chromatic root.

- ✓ Quadratic Integers
- ✓ Cubic Integers



(2,4)-biclique

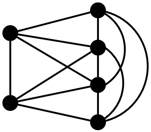


(3,4)-biclique

Bibliques

A (d, k)-biclique:

- Cliques K_d and K_k
- Some edges connecting K_d and K_k



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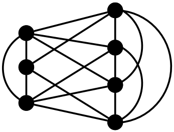
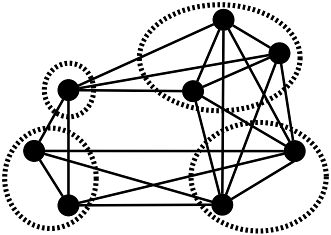


(3,4)-biclique

Ring of Cliques

The *Ring of Cliques* $R(a_1, a_2, \ldots, a_k)$:

- k cliques of size $a_1, a_2, \ldots, a_k$.
- All edges between consecutive cliques.
- Generalised Cycle.
- $(a_1 + a_2, a_3 + a_4)$-biclique ($k = 4$).



R(1,3,2,2)

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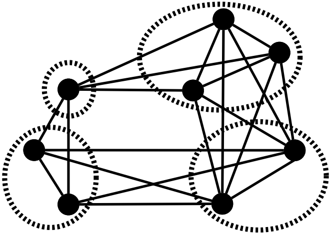

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The $alpha + n$ Conjecture

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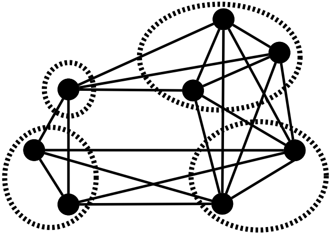

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The *Ring of Cliques* $R(a_1, a_2, \ldots, a_k)$:

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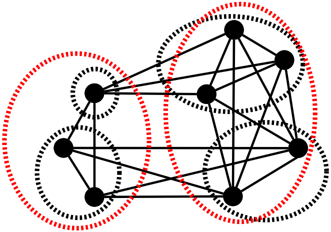


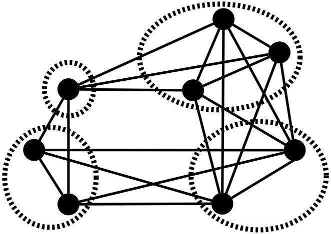
(3,5)-biclique

The *alpha* + *n* Conjecture

Quadratic Integers

The Ring of Cliques $R(1, a, b, c)$ has the *interesting factor*

$$\lambda^2 - (a + b + c)\lambda + (ab + ac + bc).$$



R(1,3,2,2)

Quadratic Integers

The Ring of Cliques $R(1, a, b, c)$ has the *interesting factor*

$$\lambda^2 - (a + b + c)\lambda + (ab + ac + bc).$$

For any quadratic algebraic integer α there is an $n \in \mathbb{N}$ such that $\alpha + n$ is a chromatic root of $R(1, a, b, c)$.

The $\alpha + n$ Conjecture

Cubic Integers
A $(3, k)$-biclique has a cubic interesting factor with six parameters.

For any cubic algebraic integer α there is an $n \in \mathbb{N}$ such that $\alpha + n$ is a chromatic root of a $(3, k)$-biclique. (Bohn, 2014)

The $\alpha + n$ Conjecture

Algebraic Integers

Conjecture

*For any algebraic integer α of degree d there is an $n \in \mathbb{N}$ such that $\alpha + n$ is a chromatic root of a **biclique**.*

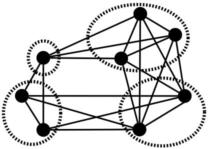
True for $d \leq 3$ and empirically true for $d = 4$.

Algebraic Integers

Conjecture ($n\alpha$ Conjecture (Cameron))

If α is a chromatic root, then so too is $n\alpha$ for all $n \in \mathbb{N}$.

- ✓ Rings of Cliques $R(1, a_1, a_2, \ldots, a_k)$.
- ✓ Generalised Theta Graph $\Theta_{a_0, a_1, \ldots, a_k}$.

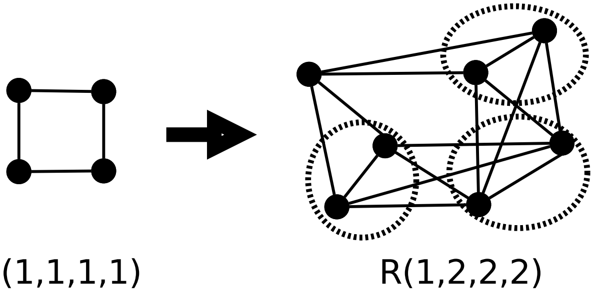


R(1,3,2,2)

The $n\alpha$ Conjecture

True for Rings of Cliques.

If α is a root of the interesting factor of $R(1, a_1, a_2, \ldots, a_k)$, then for any natural number n, $n\alpha$ is root of $R(1, a_1 n, a_2 n, \ldots, a_k n)$.

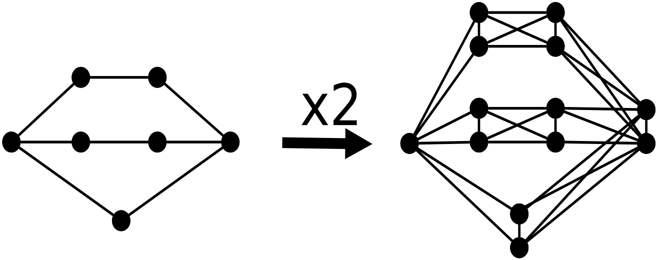


The $n\alpha$ Conjecture

True for Generalised Theta Graphs.

If α is a root of the interesting factor of $\theta_{a_1,a_2,\ldots,a_k}$, then for any natural number n, $n\alpha$ is root of a *clique theta graph*.

A *clique-theta graph* can be obtained by ”blowing up“ the vertices of a generalised theta graph into cliques.



Families of graphs

- Cycles
- Rings of cliques are generalised cycles.
- Generalised theta graphs are generalised cycles.
- Clique theta graphs are generalisations of both.

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Algebraic Integers

Question

If α is a chromatic root, is $f(\alpha)$ a chromatic root?

- $f(\alpha) = n\alpha$ (for some families of graphs)
- Others?

The Power of Alpha Conjecture

Question

If α is a chromatic root, is α^k a chromatic root?

The Power of Alpha Conjecture

Cycles

$$P(C_n;\lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1).$$

$$\downarrow$$

$$\lambda = 1 - Y$$

$$\downarrow$$

$$P(C_n;Y) = (-1)^n Y\left(Y^{n-1} - 1\right)$$

The Power of Alpha Conjecture

Cycles

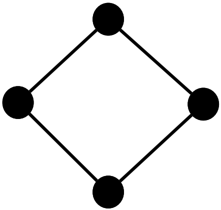
$$P(C_n;\lambda) = (\lambda-1)^n + (-1)^n(\lambda-1).$$
$$P(C_n;Y) = (-1)^n Y\left(Y^{n-1}-1\right) \text{ where } \lambda = 1-Y.$$

α root of $P(C_{4a-1};Y) \Rightarrow \alpha^2$ root of $P(C_{2a};Y)$, $a \geq 2$.

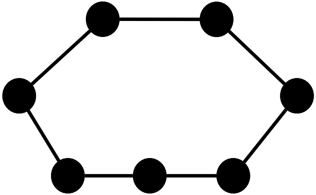
$$\begin{aligned} P(C_{4a-1};Y) &= (-1)^{4a-1} Y\left(Y^{4a-2}-1\right) \\ &= (-1)^{4a-1} Y\left((Y^2)^{2a-1}-1\right) \\ &= -\frac{P(C_{2a};Y^2)}{Y}. \end{aligned}$$

Generalised Theta Graphs

- $C_{2a} = \Theta_{a,a} \in \{\Theta_{\underbrace{a, a, \ldots, a}_{k+1}} : a \geq 2, k \geq 1\} = \Theta_{a^{k+1}}$.
- $C_{4a-1} = \Theta_{2a-1,2a} \in \{\Theta_{(k+1)a-k,\ldots,(k+1)a)} : a \geq 2, k \geq 1\} = \Theta_{c(a,k+1)}$



$\Theta_{2,2} = \Theta_{2^2}$



$\Theta_{3,4} = \Theta_{c(2,2)}$

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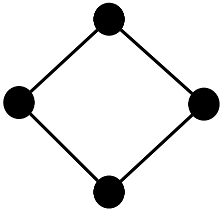

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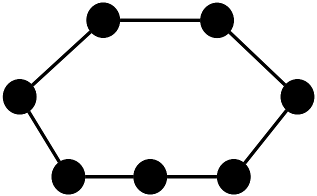

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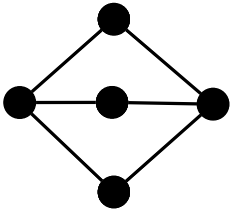
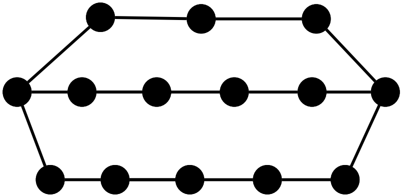


$\Theta_{2,2,2} = \Theta_{2^3}$



$\Theta_{4,5,6} = \Theta_{c(2,3)}$

Generalised Theta Graphs

Upsize

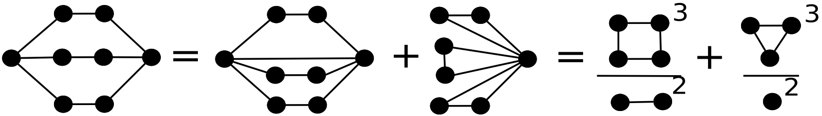
α root of $P(C_{4a-1}; Y) \Rightarrow \alpha^2$ root of $P(C_{2a}; Y)$, $a \geq 2$,

To

α root of $\Theta_{c(a,k+1)} \Rightarrow \alpha^{k+1}$ root of $\Theta_{a^{k+1}}$, $a \geq 2$.

Generalised Theta Graphs

$$
\begin{aligned}
P(\Theta_{a^{k+1}}, Y) &= \frac{P(C_{a+1}; Y)^{k+1}}{P(K_2; Y)^k} + \frac{P(C_a; Y)^{k+1}}{P(K_1; Y)^k} \\
&= \frac{\left((-1)^{a+1} Y(Y^a - 1)\right)^{k+1}}{Y^k (Y-1)^k} + \frac{\left((-1)^a (Y\left(Y^{a-1} - 1\right)\right)^{k+1}}{(1-Y)^k} \\
&= (-1)^{(a+1)(k+1)} Y(Y-1) \\
&\quad \times \frac{\left((Y^a - 1)^{k+1} - Y^k (Y^{a-1} - 1)^{k+1}\right)}{(Y-1)^{k+1}}
\end{aligned}
$$

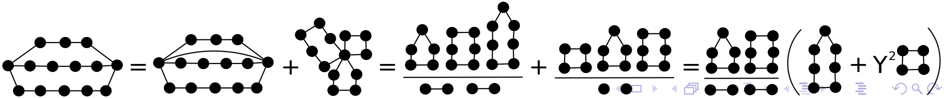


Generalised Theta Graphs

$$P(\Theta_{c(a,k+1)};Y) = \frac{P(C_{(k+1)a-k+1};Y)\dots P(C_{(k+1)a+1};Y)}{P(K_2;Y)^k} + \frac{P(C_{(k+1)a-k};Y)\dots P(C_{(k+1)a};Y)}{P(K_1;Y)^k}$$

$$= \frac{P(C_{(k+1)a-k+1};Y)\dots P(C_{(k+1)a};Y)}{P(K_2;Y)^k} \times \left(P(C_{(k+1)a+1};Y) + (-Y)^k P(C_{(k+1)a-k};Y)\right)$$

$$= \frac{P(C_{(k+1)a-k+1};Y)\dots P(C_{(k+1)a};Y)}{P(K_2;Y)^k} \times (-1)^{(k+1)a+1} Y \left(Y^{(k+1)a} - Y^{(k+1)a-1} + Y^k - 1\right)$$



Generalised Theta Graphs

- $f(Y) = (Y^a - 1)^{k+1} - Y^k(Y^{a-1} - 1)^{k+1}$

$$P(\Theta_{a^k}; Y) = (-1)^{(a+1)(k+1)} Y(Y-1)\frac{f(Y)}{(Y-1)^{k+1}}$$

- $g(Y) = Y^{(k+1)a} - Y^{(k+1)a-1} + Y^k - 1$

$$P(\Theta_{c(a,k+1)}; Y) = (-1)^{(k+1)a+1} Y \frac{P(C_{(k+1)a-k+1}; Y) \ldots P(C_{(k+1)a}; Y)}{P(K_2; Y)^k} g(Y)$$

- $\Delta\left(\frac{f(Y)}{(Y-1)^{k+1}}\right) = \Delta\left(\frac{g(Y)}{Y^{k+1}-1}\right)$

Power of α

Theorem (Delbourgo & M.)

If α is a root of $g(Y)$ then α^{k+1} is a root of $f(Y)$.

Proof.

□

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Proof.

$$\begin{aligned} f(\alpha^{k+1}) &= (\alpha^{(k+1)a} - 1)^{k+1} - (\alpha^{k+1})^k(\alpha^{(k+1)(a-1)} - 1)^{k+1} \\ &= (\alpha^{(k+1)a} - 1)^{k+1} - (\alpha^{a(k+1)-1} - \alpha^k)^{k+1}. \end{aligned}$$

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As $g(\alpha) = 0$ we have $\alpha^{(k+1)a} - 1 = \alpha^{(k+1)a-1} - \alpha^{k}$

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As $g(\alpha) = 0$ we have $\alpha^{(k+1)a} - 1 = \alpha^{(k+1)a-1} - \alpha^{k}$ and so

$$f(\alpha^{k+1}) = (\alpha^{(k+1)a-1} - \alpha^{k})^{k+1} - (\alpha^{a(k+1)-1} - \alpha^{k})^{k+1} = 0.$$

□

Generalised Theta Graphs

- $f(Y) = (Y^a - 1)^{k+1} - Y^k(Y^{a-1} - 1)^{k+1}$

$$P(\Theta_{a^k}; Y) = (-1)^{(a+1)(k+1)} Y(Y-1)\frac{f(Y)}{(Y-1)^{k+1}}$$

- $g(Y) = Y^{(k+1)a} - Y^{(k+1)a-1} + Y^k - 1$

$$P(\Theta_{c(a,k+1)}; Y) = (-1)^{(k+1)a+1} Y \frac{P(C_{(k+1)a-k+1}; Y)\ldots P(C_{(k+1)a}; Y)}{P(K_2; Y)^k} g(Y)$$

- α root of $g(Y) \Rightarrow \alpha^{k+1}$ root of $f(Y)$.

Question

Are there other families of graphs such that if α is a chromatic root, then $f(\alpha) = \alpha^{k+1}$ is a chromatic root for some $k \in \mathbb{N}$?

Generalised Theta Graphs

- $f(Y) = (Y^a - 1)^{k+1} - Y^k(Y^{a-1} - 1)^{k+1}$

$$P(\Theta_{a^k}; Y) = (-1)^{(a+1)(k+1)} Y(Y-1)\frac{f(Y)}{(Y-1)^{k+1}}$$

- $g(Y) = Y^{(k+1)a} - Y^{(k+1)a-1} + Y^k - 1$

$$P(\Theta_{c(a,k+1)}; Y) = (-1)^{(k+1)a+1} Y \frac{P(C_{(k+1)a-k+1}; Y) \ldots P(C_{(k+1)a}; Y)}{P(K_2; Y)^k} g(Y)$$

- α root of $g(Y) \Rightarrow \alpha^{k+1}$ root of $f(Y)$.

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More About Generalised Theta Graphs

- Every chromatic root λ of a generalised theta graph with $k+1$ paths lies in the disc $|Y| = |\lambda - 1| \leq k$ (Brown, 1998).
  If $|\alpha| \geq 1$, then $|\alpha| \leq |\alpha^{k+1}| \leq k$.
- Every chromatic root λ of a generalised theta graph with $k+1$ paths lies in the disc $|Y| = |\lambda - 1| \leq \frac{k}{\log 2}$ (Royle and Sokal, 2013).
  If $|\alpha| \geq 1$, then $|\alpha| \leq |\alpha^{k+1}| \leq \frac{k}{\log 2}$.
- Θ_{2^k} gives the root that maximises $|Y| = |\lambda - 1|$ over all generalised theta graphs with $k \leq 7$ paths (Brown, Hickman, Sokal and Wagner, 2000).
  (Conjectured to be true for larger k).
- If $|\alpha| \geq 1$, our result shows the chromatic root of $\Theta_{c(2,k)}$, $\alpha \leq \alpha_{k+1}$, the chromatic root of Θ_{2^k}.

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Generalised Theta Graphs

- Upsize some nice behaviour of cycles.
- Relationship between "blow-up“ factor of cliques and chromatic roots in $n\alpha$ results.
- Relationship between number of paths and chromatic roots in α^{k+1} results.
- Relationship between structure of graph and chromatic polynomial.
- Are theta graphs *special* or can we find similar results for other families of graphs?

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