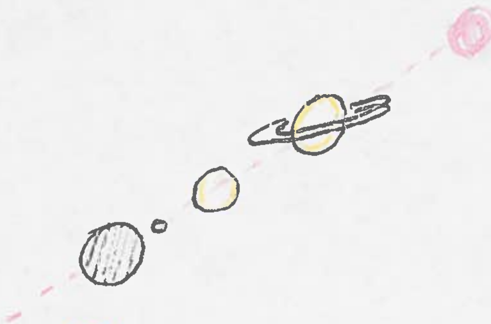
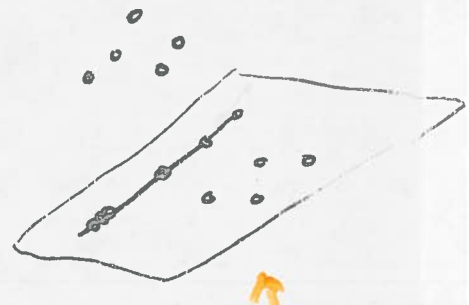




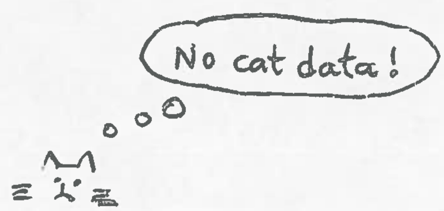
Syzygies on Tutte polynomials of freedom matroids

$$\sum_{A \subseteq E} (x-1)^{r-r(A)} (y-1)^{|A|-r(A)}$$

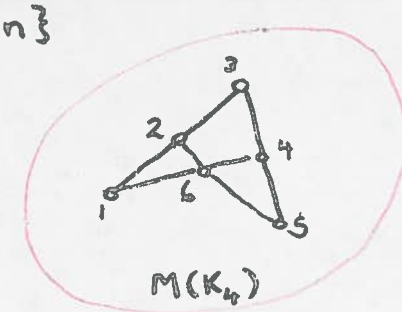
Joseph Kung

University of North Texas

His cat



G-invariant of a matroid M on $\{1,2,\cdots,n\}$



a permutation

$$r(\pi) = (r_1, r_2, \cdots, r_n)$$

rank sequence

where $r_1 = \mathrm{rank}(\pi(1))$

$r_i = \mathrm{rank}(\{\pi(1), \pi(2), \cdots, \pi(i)\}) - \mathrm{rank}(\{\pi(1), \pi(2), \cdots, \pi(i-1)\})$

$\pi = 453126 \qquad r(\pi) = 110100$

$\pi = 152346 \qquad r(\pi) = 111000$

$$G(M) = \sum_{\pi} [r(\pi)]$$

all permutations

formal symbols

$$G(M(K_4)) = 576\,[111000] + 144\,[110100]$$

Theorem. The Tutte polynomial is a specialization
evaluation
of the G-invariant

Derksen, Derksen+Fink

$G \to T$ specialization

$$Sp:\ [r_1 r_2 \cdots r_n] \mapsto \sum_{m=0}^{n} \frac{(x-1)^{r-wt(r_1 r_2 \cdots r_m)}\,(y-1)^{m-wt(r_1 r_2 \cdots r_m)}}{m!\,(n-m)!}$$

$$wt(r_1 r_2 \cdots r_m) = \#\,1\text{'s in } r_1 r_2 \cdots r_m$$

linear map:

$$\mathcal{G}(n,r) \longrightarrow \mathcal{T}(n,r)$$

vector space of formal linear combinations of symbols $[r_1 r_2 \cdots r_n]$

subspace in $\mathbb{Q}[x,y]$ spanned by Tutte polynomials of rank-r matroid on an n-set

Syzygies

ker Sp is spanned by

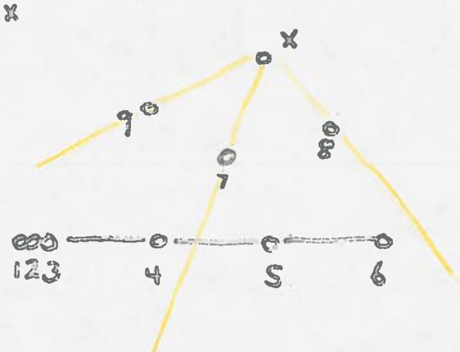
$$[-01-01-] - [-01-10-] - [-10-01-] + [-10-10-]$$

$$\begin{tikzcd}[column sep=small]
 & -10-10- \arrow[dl, no head] \arrow[dr, no head] & \\
-01-10- \arrow[dr, no head] & & -10-01- \arrow[dl, no head] \\
 & -01-01- &
\end{tikzcd}$$

Example: $[1\underline{0101}00] - [1\underline{0110}00] - [1\underline{1001}00] + [1\underline{1010}00]$

Freedom matroids

F(1001001100)
1 2 3 4 5 6 7 8 9 x



in 3-dimension space

Theorem. The G-invariants of freedom matroids is a basis for $\mathcal{G}(n,r)$. Thus, every Tutte polynomial is a linear combination of TP's of freedom matroids in a "natural" way.

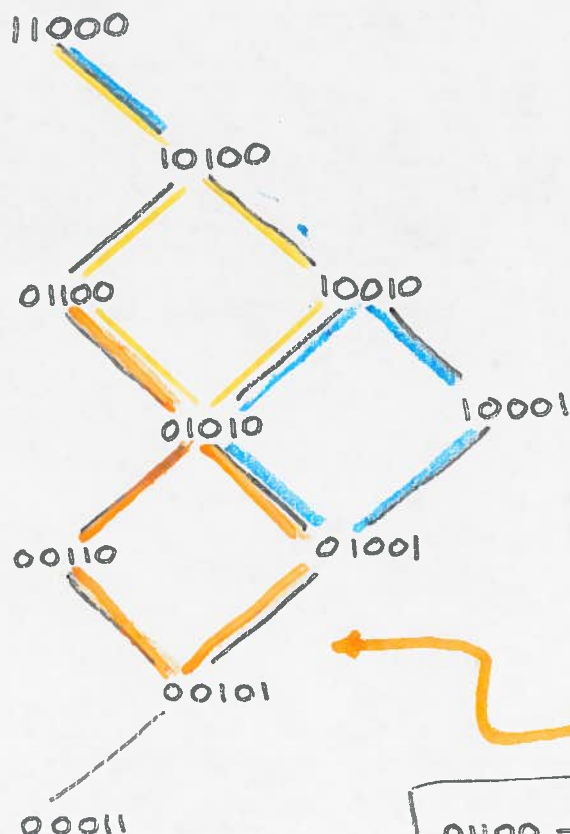
Linear relations on TP's of freedom matroids:

$\trianglerighteq$ - partial order

1011000010 $\trianglerighteq$ 1001100001

1's are moved to the left

$\trianglerighteq$ = weak order on freedom matroid
= a sublattice of Young's partition lattice



linear relation on TP's

$$\boxed{\begin{aligned} &01100 - 01010 \\ &\quad = 01010 - 00110 - 01001 + 00101 \end{aligned}}$$

Weak order on freedom matroids, $r=2$, $n=5$

How good is G or TP at distinguishing matroids?

0th order: $\mathcal{G}(M_1) = \mathcal{G}(M_2)$ [hence $T(M_1) = T(M_2)$]

"Almost all" matroids: paving matroids, ...

1th order: $\mathcal{G}(M_1) \neq \mathcal{G}(M_2)$, $T(M_1) = T(M_2)$

Essentially only two examples; these imply linear relations between TP's of freedom matroids

2nd order: $T(M_1) \neq T(M_2)$

Freedom matroids, despite linear dependence

Areas of ignorance / research:

Do this on minor-closed classes ... are there analogues of freedom matroids for graphs, binary matroids, ... ?

dim (span TP's of all rank-r matroids on an n-set)

$= r(n-r)+1$

dim (span ... of graphic ....) = ?

binary

⋮