Critical manifolds and exact solvability from a topologically weighted Tutte polynomial

Jesper L. Jacobsen [1,2]

[1]Laboratoire de Physique Théorique, École Normale Supérieure, Paris

[2]Université Pierre et Marie Curie, Paris

Tutte Polynomial Workshop, Royal Holloway, 14 July 2015

Collaborators

- Christian R. Scullard (Lawrence Livermore National Lab.)
- Masayuki Ohzeki (Kyoto Univ.)
- Tony Guttmann (Melbourne Univ.)

Introduction

Study of critical phenomena

Given a (two-dimensional) lattice model, one aims at determining:

1. Critical temperature T_c (lattice specific)
2. Critical exponents (universal)
3. All corrections to scaling (CFT characters)
4. Exact correlation functions (Bethe Ansatz, SLE,...)

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Exceptions are very particular models on very particular lattices

- Known T_c seem to be given by simple algebraic curves
- How to determine critical manifolds of standard models (Potts, site percolation, O(n),...) on an arbitrary lattice?

Potts model and bond percolation

Partition function

$$Z = \sum_{\sigma} \prod_{(ij)\in E} \exp\left(K\delta_{\sigma_i,\sigma_j}\right)$$

- Spins $\sigma_i = 1, 2, \ldots, q$ with nearest-neighbour coupling K
- Planar lattice $G = (V, E)$ with vertices $i \in V$ and edges $(ij) \in E$

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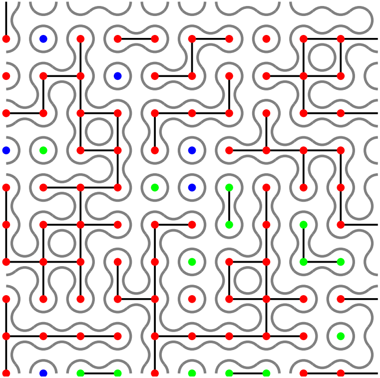
Fortuin-Kasteleyn representation alias Tutte polynomial

- Write $\exp\left(K\delta_{\sigma_i,\sigma_j}\right) = 1 + v\delta_{\sigma_i,\sigma_j}$ with $v := \mathrm{e}^K - 1$

$$Z = \sum_{A\subseteq E} v^{|A|} q^{k(A)}$$

- $q \to 1$ produces bond percolation, with $p = \frac{v}{1+v}$

Example of configuration for $q = 3$



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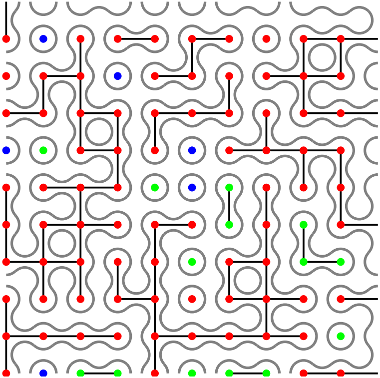

Colours: Potts spins σ_i
Black bonds: Fortuin-Kasteleyn (FK) clusters
Gray curves: Equivalent surrounding loops

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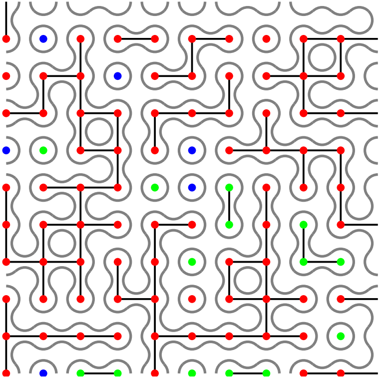

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Spins on each FK cluster are independent (weight q per cluster)

Critical manifold and percolation threshold

Solvability only on a few lattices G

$$
\begin{aligned}
(v^2 - q)(v^2 + 4v + q) &= 0\,, && \text{(square lattice)} \\
v^3 + 3v^2 - q &= 0\,, && \text{(triangular lattice)} \\
v^3 - 3qv - q^2 &= 0\,. && \text{(hexagonal lattice)}
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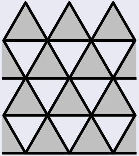
All solvable cases are of the “3-terminal form”



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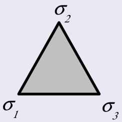
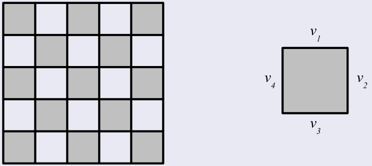


(b)

Archimedean lattices

All Archimedean lattices can be written in “4-terminal form”



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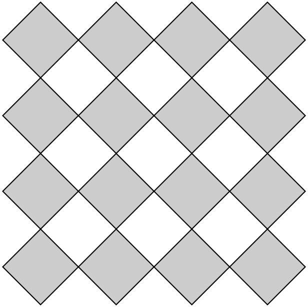

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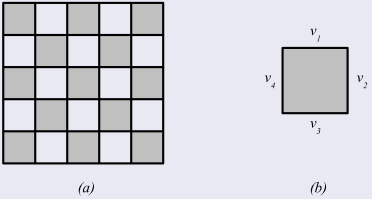

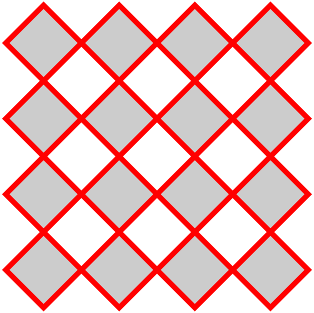

Square lattice

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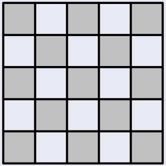

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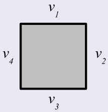

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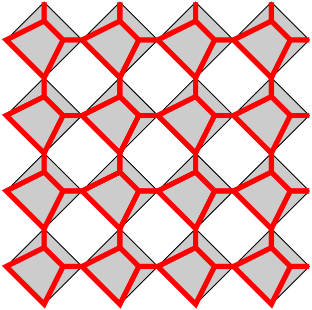

$(4, 8^2)$ lattice

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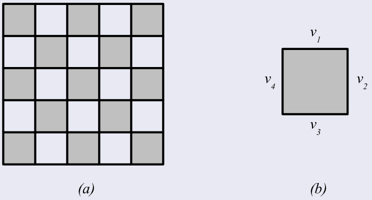

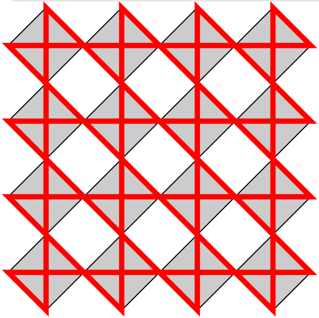

Kagome lattice

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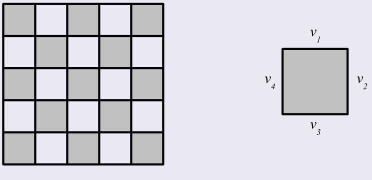

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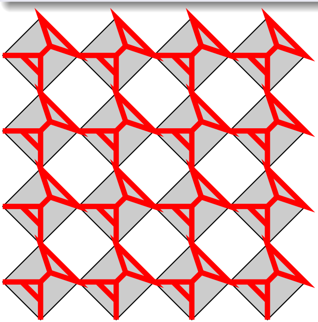
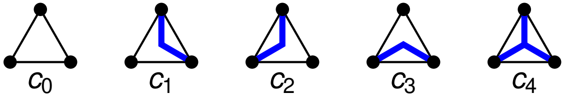


$(3, 12^2)$ lattice

- Boltzmann weight of elementary triangle

$$w_{123} = c_0 + c_1\delta_{23} + c_2\delta_{13} + c_3\delta_{12} + c_4\delta_{123}$$



Solvability of 3-terminal lattices

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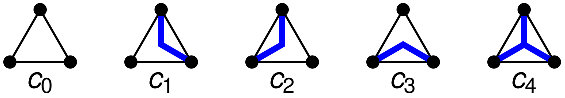

- FK cluster partition function $Z = \sum_{A\subseteq E} q^{k(A)} \prod_{p=0}^{4} (c_p)^{N_p}$

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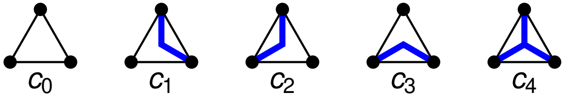

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Solution for the critical manifold (Wu-Lin 1980)

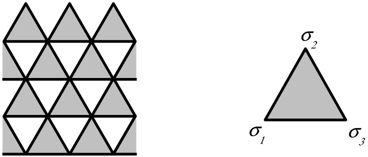
$$P(q,v) = c_4 - qc_0 = 0$$

Proof 1: $\frac{\pi}{3}$ rotation symmetry of equivalent loop model
Proof 2: Fixed point of combined duality and decimation transform

The graph polynomial

Terminology

- Let B (the “basis”) be a finite portion of G with N terminals
- G is obtained by tiling space with B in a certain way (the “embedding”), gluing copies of B at the terminals



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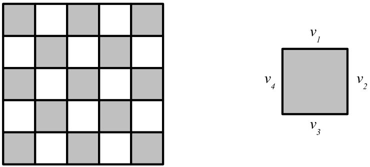

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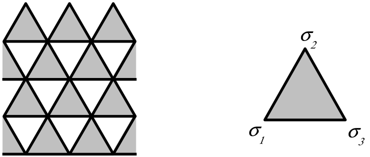

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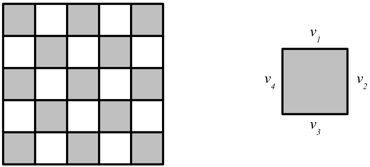

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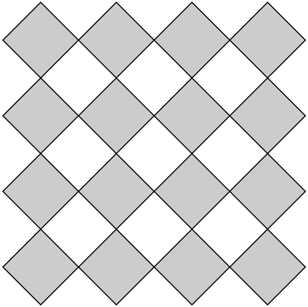
Definition (only and hence most important in this talk!)

$$P_B(q, v) = Z_{\rm 2D} - qZ_{\rm 0D}$$

Example: Square lattice with $B =$ square of four edges



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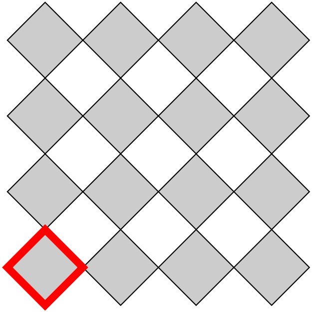

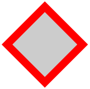

qv^4

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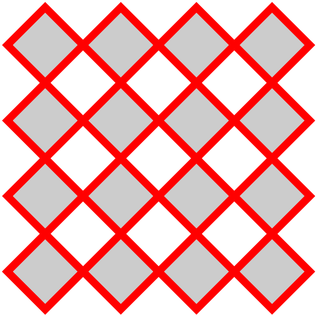

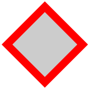

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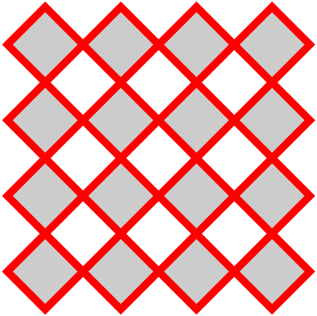

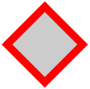

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$Z_{2\text{D}} = qv^4$

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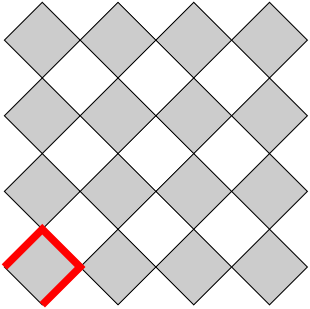

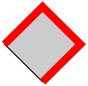

$4qv^3$

$Z_{\mathrm{2D}} = qv^4$

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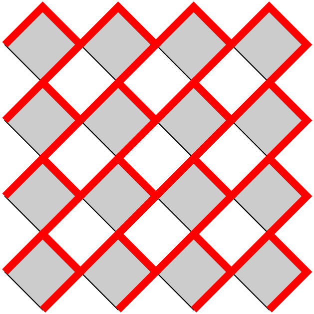

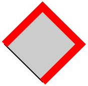

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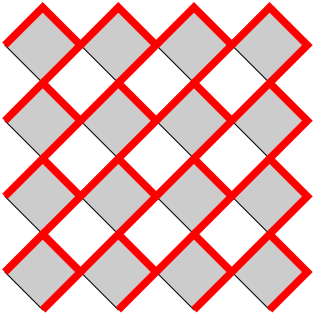

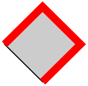

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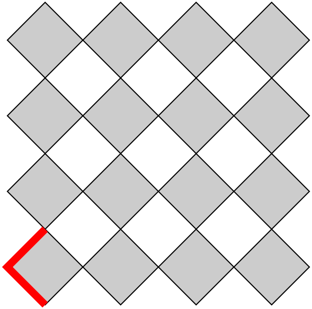

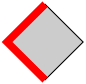

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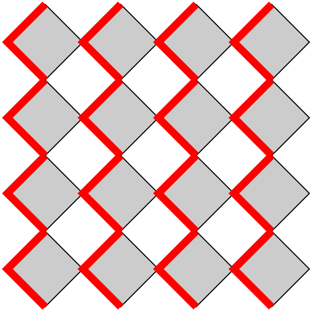

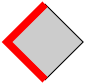

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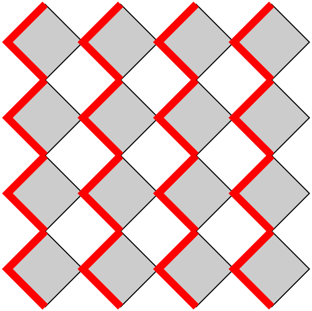

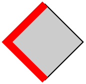

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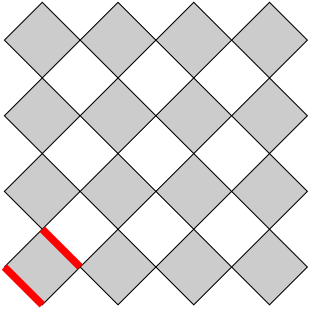

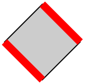

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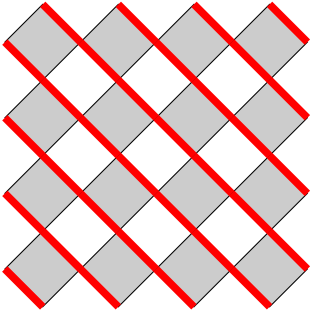

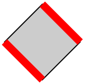

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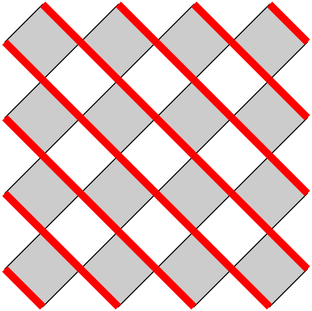

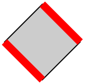

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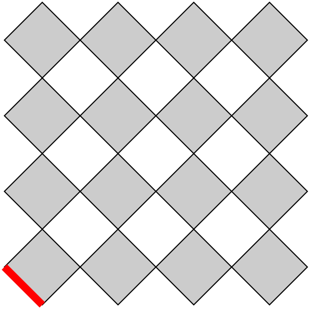

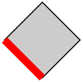

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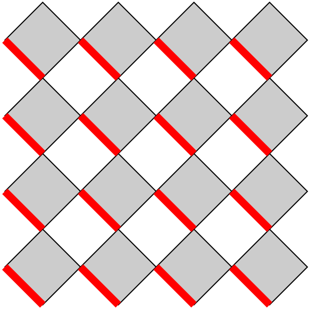

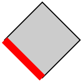

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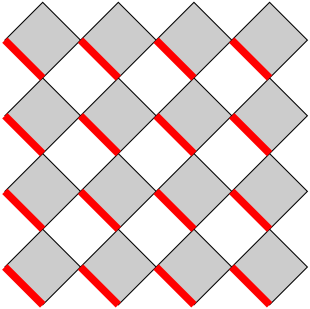

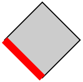

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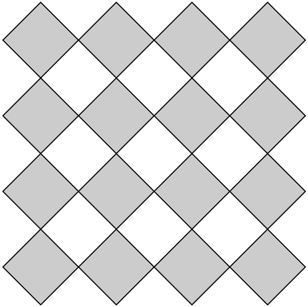
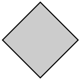
$$Z_{\rm 0D} = 4qv$$

Example: Square lattice with B = square of four edges





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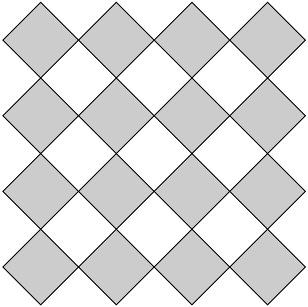

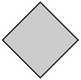

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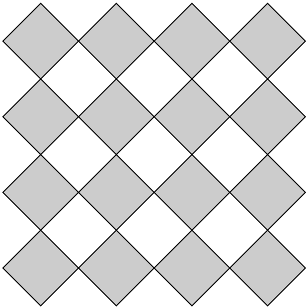


$$Z_{2\mathrm{D}} = qv^4 + 4qv^3$$

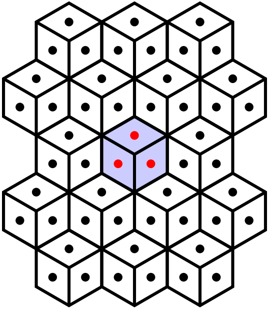
$$Z_{1\mathrm{D}} = 4qv^2 + 2qv^2$$

$$Z_{0\mathrm{D}} = 4qv + q^2$$

$$P_B(q,v) = Z_{2\mathrm{D}} - qZ_{0\mathrm{D}} = q(v^2 - q)(v^2 + 4v + q)$$

Example: Site percolation on kagome lattice

籠目



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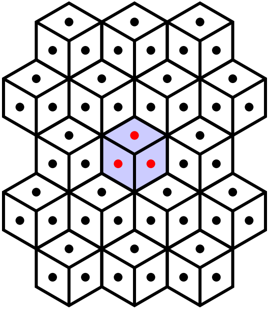

$$
\begin{aligned}
P(2D;B) &= p^3\,, \\
P(1D;B) &= 3p^2(1-p)\,, \\
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\end{aligned}
$$

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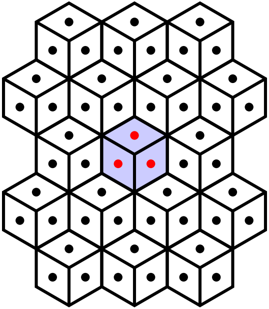

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$P_B(p) = 1 - 3p^2 + p^3 = 0$

$p_c = 1 - 2\sin(\pi/18) = 0.652\,704\cdots$ provides the exact threshold.

General structure of the results (1)

Factorisation property for $G =$ solvable case

- $P_B(q, v)$ factorises over the integers, shedding a “small factor”
- Small factor independent of size of B, and gives exact solution
- Checked for “all” known solutions of the Potts model

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Computing $P_B(q, v)$ serves to detect exact solvability.

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∞	$0.740\,420\,798\,850\,9(4)$	[Bulirsch-Stoer extrapolation]
Numerics	$0.740\,420\,77\ (2)$	[Ding et al. 2010]

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- With extrapolation, we can attain 13-digit precision
- Same precision for other q, at least when $v > 0$

Site percolation on square lattice

1	$0.500\,000\,000\,000\,000\cdots$
2	$0.541\,196\,100\,146\,196\cdots$
3	$0.586\,511\,455\,112\,675\cdots$
4	$0.590\,672\,112\,331\,028\cdots$
5	$0.591\,988\,256\,518\,333\cdots$
6	$0.592\,395\,070\,817\,704\cdots$
7	$0.592\,561\,037\,427\,664\cdots$
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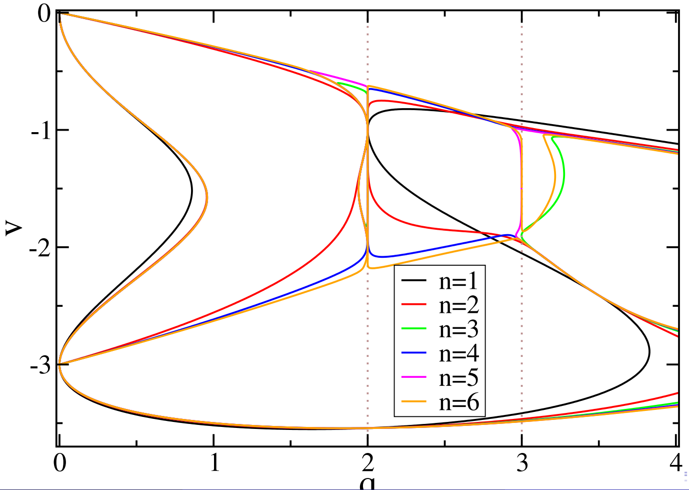
- Method not as precise as for bond percolation
- Convergence $\sim 1/n^4$ for site and $\sim 1/n^6$ for bond percolation

Example of complete phase diagram: Kagome lattice

For clarity we show only the antiferromagnetic region ($v < 0$):



Get $P_B(q, v)$ by solving an eigenvalue problem

3 transfer matrix methods for $P_B(q, v) = Z_{\rm 2D} - qZ_{\rm 0D}$ on $L \times M$ basis:

1. Build partition functions conditioned on the FK connectivities among all $2(L + M)$ boundary spins. Then construct $Z_{\rm 2D}$ and $Z_{\rm 0D}$ by some gluing and an Euler-like identity.

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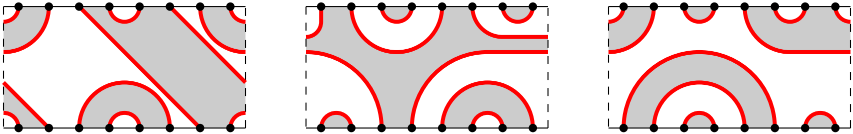
3. Take $M \to \infty$. When $n_{\rm wind} = 0$, two isomorphic representations ('open' and 'closed') of the periodic Temperley-Lieb algebra. Computing their eigenvalues Λ requires states on only L spins. Then $Z_{\rm 2D} \sim (\Lambda^{\rm max}_{\rm open})^M$ and $Z_{\rm 0D} \sim (\Lambda^{\rm max}_{\rm closed})^M$, so:

$$P_B(q, v) = 0 \quad \Leftrightarrow \quad \Lambda^{\rm max}_{\rm open} = \Lambda^{\rm max}_{\rm closed}$$

A few pictures to illustrate those ‘open’ and ‘closed’ representations:



(a) (b) (c)

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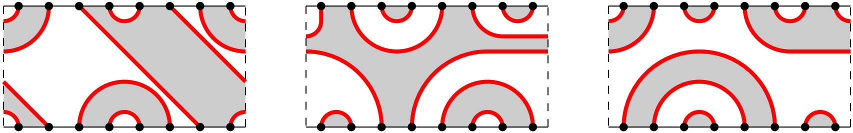

(a) (b) (c)

- We can discard representations with through-going lines (a).
- Gluing of two open representations (b).
- Gluing of two closed representations (c).

Bond percolation on kagome lattice with $L \times \infty$ bases

1 $\quad 0.524\,{\color{red}4}29\,717\,521\,274\,793 \cdots$

Bond percolation on kagome lattice with $L \times \infty$ bases

1	0.524 429 717 521 274 793 ⋯
2	0.524 406 057 896 062 634 ⋯

Bond percolation on kagome lattice with $L \times \infty$ bases

1	$0.524\,429\,717\,521\,274\,793\cdots$
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12	$0.524\,404\,999\,180\,248\,443 \cdots$
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14	$0.524\,404\,999\,172\,242\,908 \cdots$

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12	0.524 404 999 180 248 443 ···	
13	0.524 404 999 175 132 845 ···	
14	0.524 404 999 172 242 908 ···	
∞	0.524 404 999 167 439(4)	[Extrapolation]
Numerics	0.524 404 99(2)	[Feng et al. 2008]

Extrapolations

- Raw data has 11 correct digits.
- We get 14 digits after extrapolation.
- Convergence is like A/n^{α} with $\alpha = 6.00(5)$.

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- Example: Site percolation on the square lattice.
- 21 terms of raw data, giving 6 correct digits.
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- Assuming $1/n^{2k}$ with $k = 2, 3, \ldots$ we get 14 digits: $p_c = 0.592\,746\,050\,792\,10(2)$
- 10^6 times more precise than numerics: $p_c = 0.592\,746\,05(3)$

Applications to quenched random systems

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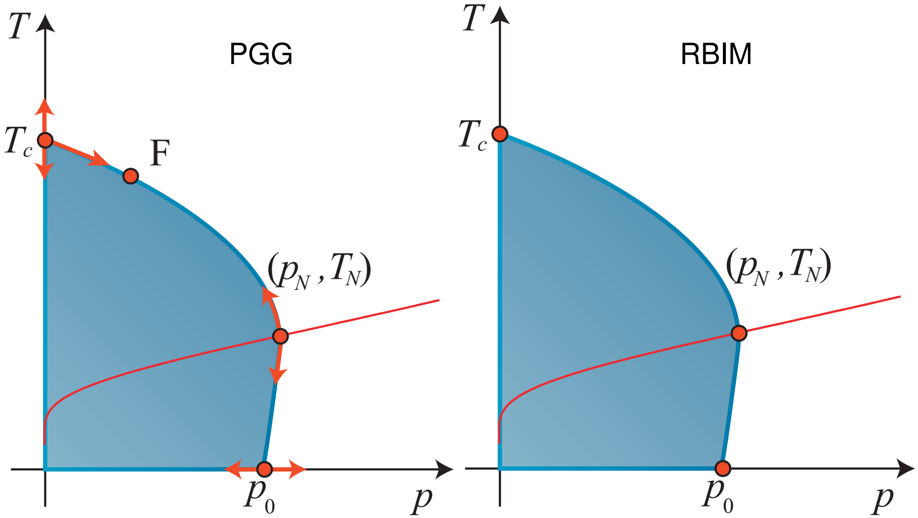
q-state Potts gauge glass (PGG)

Each bond carries a random twist τ_{ij}.

$$\begin{aligned} H &= -K \sum_{\langle ij \rangle} \delta_q(S_i - S_j + \tau_{ij}) \\ P(\tau_{ij}) &= (1-(q-1)p)\delta(\tau_{ij}) + p \sum_{\tau=1}^{q-1} \delta(\tau_{ij}-\tau) \end{aligned}$$

Schematic phase diagrams in $d = 2$



Nishimori line $e^K = \frac{p}{1-(q-1)p}$ has special gauge symmetry ($T \equiv 1/K$)

Translate $P_B(q, v)$ from FK to spin representation

Expression in terms of twisted (topological) partition functions

$$P_B(q, v) = qZ_{0,0} - \sum_{\tau_x,\tau_y=0}^{q-1} Z_{\tau_x,\tau_y} \,,$$

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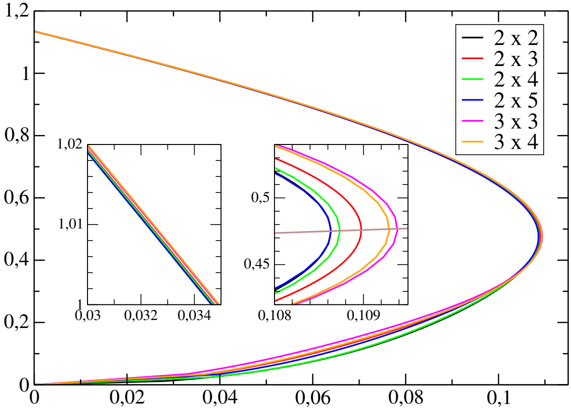
Quenched average from $n \to 0$ replica trick

$$\left[\log \left(\sum_{\tau_x,\tau_y=0}^{q-1} Z_{\tau_x,\tau_y} \right) \right] - \left[\log Z_{0,0} \right] = \log q \,,$$

where $[\cdots]$ is average over the randomness (τ_{ij}) with measure $P(\tau_{ij})$.

Ferro-para phase boundary (p, T) plane from $P_B(q, v)$

RBIM:



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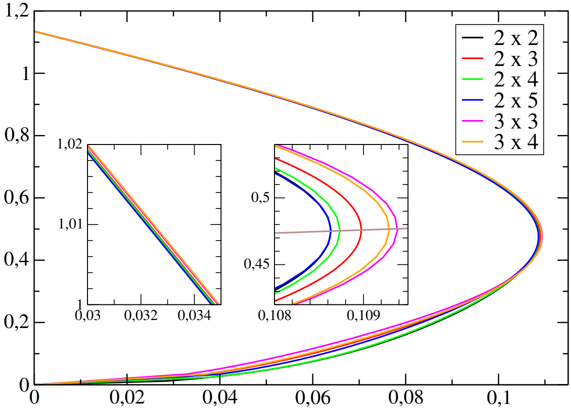

We find the Nishimori point at $p_N = 0.10929 \pm 0.00002$

Eigenvalue criterion for O(n) type models

Revisit the eigenvalue method for the Potts model

- $Z_{\rm 2D}$ (resp. $Z_{\rm 0D}$) imposes the existence of an FK (resp. dual FK) cluster spanning the cylinder.
- By CFT, the corresponding free energies satisfy, at $T = T_{\rm c}$:

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f_{\rm 2D} = -\frac{1}{L} \log Z_{\rm 2D} &= f_\infty - \frac{2\pi x_{\rm mag}}{L^2} + o(L^{-2})\,, \\
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So $f_{\rm 2D} - f_{\rm 0D} = o(L^{-2})$ at $T = T_{\rm c}$

- Setting instead $f_{\rm 2D} - f_{\rm 0D} = 0$ and solving for $T = T_{\rm c}(L)$ we can expect that $T_{\rm c}(L) = T_{\rm c} + O(L^{-\alpha})$ with α “large”.
- We found $\alpha = 4$ (resp. $\alpha = 6$) for site (resp. bond) percolation.

Try a similar approach for the O(n) model

- Find two “topological” excitations in the transfer matrix picture that lead to the *same* critical exponent x.
- Take that x to be the smallest possible (like $x_{\rm mag}$ for Potts):
  1. One-leg sector, with loop strand propagating along the cylinder.
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Coulomb gas computation: $n'_{\rm wind} = \pm\sqrt{2-n}$ gives same x

- A miracle happens: For solvable models $f_1 = f_0(n'_{\rm wind})$ *exactly* at the critical point for *any* L.
- True for integrable O(n) model on hexagonal lattice, and also for the square-lattice model with arbitrary spectral parameters.
- Take plus (resp. minus) sign in $n'_{\rm wind}$ for dense (resp. dilute) phase.
- Some new kind of “discrete holomorphicity”?

Now use as an approximate method for non-integrable models

- Equally-weighted polymers on square lattice (weight z per step).
- This is the model studied by Guttmann et al. by series techniques.
- Our final extrapolated result reads: $z_c = 0.3790522777533(2)$. Series [Clisby-Jensen 2011] yield: $z_c = 0.379052277752(3)$.
- And on the triangular lattice: $z_c = 0.2409175743991(1)$ Series [Jensen 2004] result: $z_c = 0.2409175745(15)$.
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We do not (yet) know the equivalent of this on finite bases. The two transfer matrix sectors are asymptotically of the same size, but *not* isomorphic this time. So the criterion cannot be as simple as $Z_{\rm 2D} = qZ_{\rm 0D}$.

Conclusion

Graph polynomial $P_B(q, v)$

- $P_B(q, v)$ provides new method of determining critical manifolds
  - Easy to compute by hand for small bases
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Applications to quenched random systems

- High accuracy for critical point in systems with gauge symmetry
  - In progress: Expression in terms of Lyapunov exponents