Graph polynomials by counting graph homomorphisms

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Chromatic polynomial

Definition by evaluations at positive integers

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$$uv \in E(G), \quad P(G;k) = P(G\backslash uv;k) - P(G/uv;k)$$

Independence polynomial

Definition by coefficients

$$I(G;x) = \sum_{1 \leq j \leq |V(G)|} b_j(G) x^j,$$

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$$v \in V(G), \qquad I(G;x) = I(G-v;x) + xI(G-N[v];x)$$

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$f : V(G) \to V(H)$ is a *homomorphism* from G to H if
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$$\hom(G, H) = \sum_{f:V(G)\to V(H)} \prod_{uv\in E(G)} a_{f(u),f(v)}.$$

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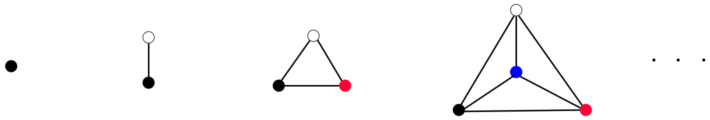
$$\begin{aligned}\hom(G,H) &= \#\{\text{homomorphisms from } G \text{ to } H\} \\ &= \#\{H\text{-colourings of } G\}\end{aligned}$$

when H simple $(a_{s,t} \in \{0,1\})$ or multigraph $(a_{s,t} \in \mathbb{N})$

Example 1



(K_k)

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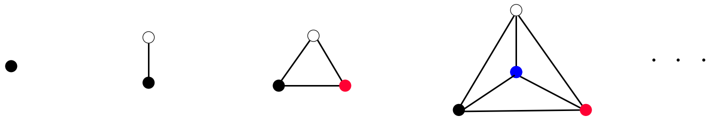

$$(K_k)$$

$$\hom(G, K_k) = P(G; k)$$

chromatic polynomial

The main question

Which sequences $(H_{k,\ell,\dots})$ of simple graphs are such that, for all graphs G, for each $k, \ell, \dots \in \mathbb{N}$ we have

$$\hom(G, H_{k,\ell,\dots}) = p(G; k, \ell, \dots)$$

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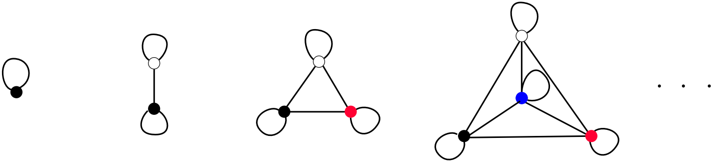
Characterizing simple graph sequences $(H_{k,\ell,\ldots})$ with this property gives straightforward characterization for multigraph sequences too (allowing multiple edges & loops).

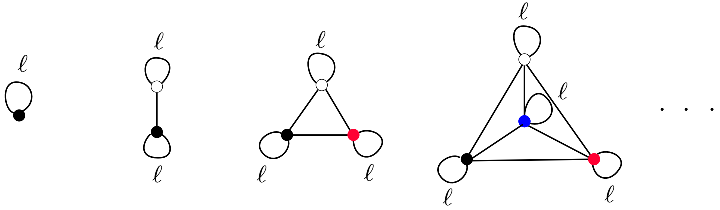
Example 2: add loops



$$(K_k^1)$$

$$\hom(G, K_k^1) = k^{|V(G)|}$$

Example 3: add ℓ loops



$$(K_k^\ell)$$

$$\hom(G, K_k^\ell) = \sum_{f:V(G)\to[k]} \ell^{\#\{uv\in E(G)\,|\,f(u)=f(v)\}}$$

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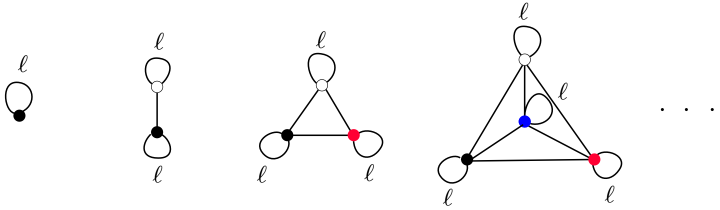

(K_k^ℓ)

$$\hom(G, K_k^\ell) = \sum_{f:V(G)\to[k]} \ell^{\#\{uv\in E(G)\,|\,f(u)=f(v)\}}$$

$$= k^{c(G)}(\ell-1)^{r(G)}\,T(G; \tfrac{\ell-1+k}{\ell-1}, \ell)$$ (Tutte polynomial)

Example 4

$$(K_1^1 + K_{1,k})$$

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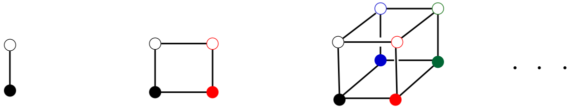
$$\hom(G, K_1^1 + K_{1,k}) = I(G; k)$$

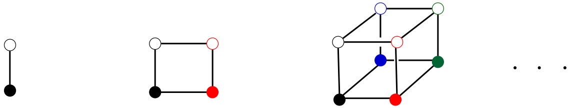
independence polynomial

Example 5



$$(K_2^{\square k}) = (Q_k) \text{ (hypercubes)}$$

Example 5

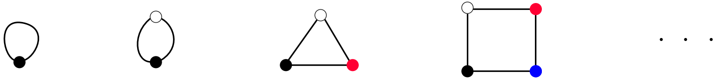


$(K_2^{\square k}) = (Q_k)$ *(hypercubes)*

Proposition (Garijo, G., Nešetřil, 2013+)

$\hom(G, Q_k) = p(G; k, 2^k)$ *for bivariate polynomial* $p(G)$

Example 6



(C_k)

$\hom(C_3, C_3) = 6$, $\hom(C_3, C_k) = 0$ when $k = 2$ or $k \geq 4$

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(H_k) is *strongly polynomial* (in k) if $\forall G \; \exists$ polynomial $p(G)$ such that $\hom(G, H_k) = p(G; k)$ for all $k \in \mathbb{N}$.

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- (K_k^ℓ) is strongly polynomial (in k, ℓ)

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Proposition (de la Harpe & Jaeger 1995)

Simple graphs (H_k) form strongly polynomial sequence $\iff$ $\forall$*connected S* $\#\{\text{induced subgraphs} \cong S \text{ in } H_k\}$ *polynomial in k*

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Generalized Johnson graph $J_{k,r,D}$, $D \subseteq \{0, 1, \ldots, r\}$
vertices $\binom{[k]}{r}$, edge uv when $|u \cap v| \in D$

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- Kneser graphs $D = \{0\}$ $K_{k:r}$

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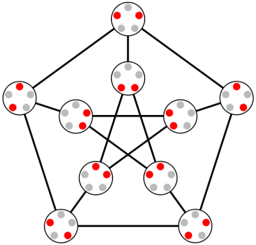
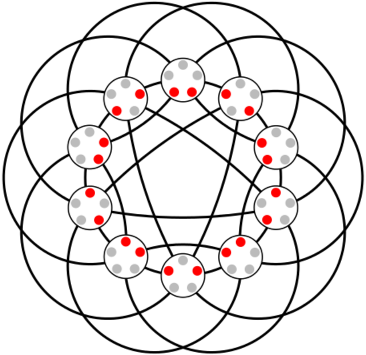


Petersen graph $= K_{5:2}$

Figure by Watchduck (a.k.a. Tilman Piesk), Wikimedia Commons



Johnson graph $J(5,2)$

Figure by Watchduck (a.k.a. Tilman Piesk). Wikimedia Commons

Fractional chromatic number of graph G:

$$\chi_f(G) = \inf\{\frac{k}{r} : k, r \in \mathbb{N},\ \hom(G, K_{k:r}) > 0\},$$

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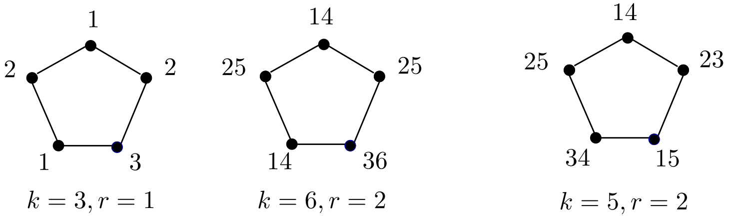
$$\chi_f(G) = \inf\{\frac{k}{r} : k, r \in \mathbb{N},\ \hom(G, K_{k:r}) > 0\},$$

For $k \geq 2r$, $\chi(K_{k:r}) = k - 2r + 2$, while $\chi_f(K_{k:r}) = \frac{k}{r}$

Fractional colouring example: C_5 to $K_{k:r}$



$\chi(C_5) = 3$ but by the homomorphism from C_5 to Kneser graph $K_{5:2}$ (Petersen graph) $\chi_f(C_5) \leq \frac{5}{2}$ (in fact wih equality)

Proposition

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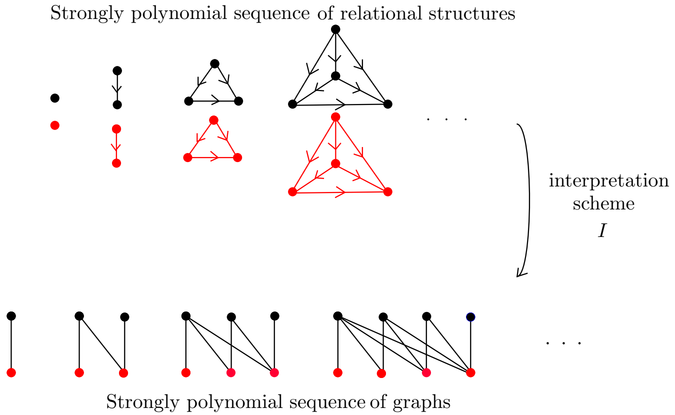
Problem

Interpret $\binom{k}{r}^{-c(G)} \hom(G, J_{k,r,D})$ in terms of the cycle matroid of G alone. In particular, what is its evaluation at $k = -1$ (acyclic orientations for the chromatic polynomial $= 1, D = \{0\}$).

Construction [G., Nešetřil, Ossona de Mendez 2014+]

Strongly polynomial sequences by quantifier-free (QF) interpretation of relational structures.



Satisfaction sets

Quantifier-free formula ϕ with n free variables ($\phi \in \mathrm{QF}_n$) with symbols from relational structure $\mathbf{H}$ with domain $V(\mathbf{H})$.

Satisfaction set $\phi(\mathbf{H}) = \{(v_1, \ldots, v_n) \in V(\mathbf{H})^n : \mathbf{H} \models \phi\}$.

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e.g. for graph structure H (symmetric binary relation $x \sim y$ interpreted as x adjacent to y), and given graph G on n vertices,

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$$|\phi_G(H)| = \hom(G, H).$$

Strongly polynomial sequences of structures

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$\forall \phi \in QF \, \exists$ polynomial $r(\phi) \, \forall k \in \mathbb{N} \quad |\phi(\mathbf{H}_k)| = r(\phi; k)$

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Lemma

Equivalently,

- $\forall \mathbf{G}\ \exists$ *polynomial* $p(\mathbf{G})\ \forall k \in \mathbb{N} \quad \hom(\mathbf{G}, \mathbf{H}_k) = p(\mathbf{G}; k)$, *or*
- $\forall \mathbf{F}\ \exists$ *polynomial* $q(\mathbf{F})\ \forall k \in \mathbb{N} \quad \mathrm{ind}(\mathbf{F}, \mathbf{H}_k) = q(\mathbf{F}; k)$.

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Transitive tournaments $(\mathbf{T}_k)$ strongly polynomial sequence of digraphs (e.g. count induced substructures).

Graphical QF interpretation schemes

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Definition (Graphical QF interpretation scheme)

Exponent $p \in \mathbb{N}$, formula $\iota \in \mathrm{QF}_p(\sigma)$ and symmetric formula $\rho \in \mathrm{QF}_{2p}(\sigma)$.
For every σ-structure $\mathbf{A}$, the interpretation $I(\mathbf{A})$ has

$$\text{vertex set} \quad V = \iota(\mathbf{A}),$$

$$\text{edge set} \quad E = \{\{\mathbf{u}, \mathbf{v}\} \in V \times V : \ \mathbf{A} \models \rho(\mathbf{u}, \mathbf{v})\}.$$

Graphical QF interpretation schemes

Example

- (Complementation) $p = 1$, $\iota = 1$ (constantly true), $\rho(x, y) = \neg R(x, y)$ ($R(x, y)$: adjacency between x and y).

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- (Complementation) $p = 1$, $\iota = 1$ (constantly true),
  $\rho(x, y) = \neg R(x, y)$ ($R(x, y)$: adjacency between x and y).
- (Square of a graph) $p = 1$, $\iota = 1$, and
  $\rho(x, y) = R(x, y) \vee (\exists z\, R(x, z) \wedge R(z, y))$
  (requires a quantifier, so not a QF interpretation scheme).

Graphical QF interpretation schemes

$$I: \text{Relational } \sigma\text{-structures } \mathbf{A} \longrightarrow \text{Graphs } H$$

Lemma

There is

$$\widetilde{I}: \phi \in \mathrm{QF}(\mathrm{Graphs}) \longmapsto \widetilde{I}(\phi) \in \mathrm{QF}(\sigma\text{-structures})$$

such that

$$\phi(I(\mathbf{A})) = \widetilde{I}(\phi)(\mathbf{A})$$

In particular, $(\mathbf{A}_k)$ strongly polynomial $\Rightarrow$ $(H_k) = (I(\mathbf{A}_k))$ strongly polynomial.

From graphs to graphs

- All previously known constructions of strongly polynomial graph sequences (complementation, line graph, disjoint union, join, direct product,...) special cases of interpretation schemes I from Marked Graphs (added unary relations) to Graphs.

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- Cartesian product and other more complicated graph products are special kinds of such interpretation schemes too.

Example

(Cartesian product of graphs G_1 and G_2)

$$\mathbf{A} = G_1 \sqcup G_2$$

$$U_i(v) \quad \Leftrightarrow \quad v \in V(G_i),$$

$$R_i(u,v) \quad \Leftrightarrow \quad uv \in E(G_i) \quad (i = 1, 2)$$

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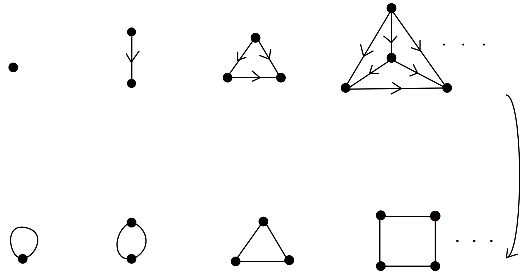
Interpretation scheme I of exponent $p = 2$ defined on (U_1, U_2, R_1, R_2)-relational structures $\mathbf{A}$ by

$$\iota(x_1, x_2) : U_1(x_1) \wedge U_2(x_2)$$

$$\rho(x_1, x_2, y_1, y_2) : [R_1(x_1, y_1) \wedge (x_2 = y_2)] \vee [(x_1 = y_1) \wedge R_2(x_2, y_2)]$$

Require quantifier-free interpretation



edge uv when

$$(u \to v) \wedge \neg \exists w (u \to v \wedge w \to v)$$
$$\vee \quad \forall w (u \to w \wedge w \to v)$$

or the same formula
with u and v swapped

Cycles (C_k) from tournaments $\mathbf{T}_k$ require quantification.
Sequence (C_k) is not strongly polynomial.

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Generalized Johnson graphs $(J_{k,r,D})$ are QF interpretations of transitive tournaments:

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Generalized Johnson graphs $(J_{k,r,D})$ are QF interpretations of transitive tournaments:
$\mathbf{A}_k = \mathbf{T}_k$, vertices $[k]$, arcs defined by relation R.
For fixed integer r and subset $D \subseteq [r]$,

$$\iota(x_1, \ldots, x_r) : \bigwedge_{i=1}^{r-1} R(x_i, x_{i+1}) \qquad \text{vertices } r\text{-subsets of } [k]$$

$$\rho(x_1, \ldots, x_r, y_1, \ldots, y_r) : \bigvee_{\substack{I,J \subseteq [r] \\ |I|=|J| \\ |I| \in D}} \left(\bigwedge_{i \notin I, j \notin J} \neg(x_i = y_j) \wedge \bigwedge_{i \in I} \bigvee_{j \in J} (x_i = y_j) \right)$$

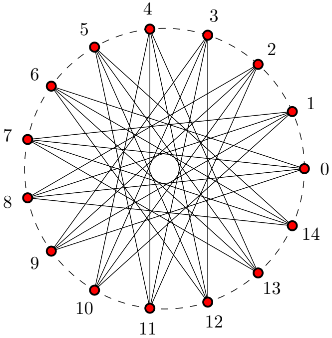
edge when subset intersection size is in D

Corollary is our previous:

Proposition

For every r, D, sequence $(J_{k,r,D})$ is strongly polynomial (in k).

$K_{s/r}$ $(s \geq 2r)$ circulant graph with vertex set $\mathbb{Z}_s = \{0, 1, \ldots, s-1\}$, vertices x, y adjacent if $x - y \in \{r, r+1, \ldots, s-r\}$. $(K_{s/1} = K_s)$



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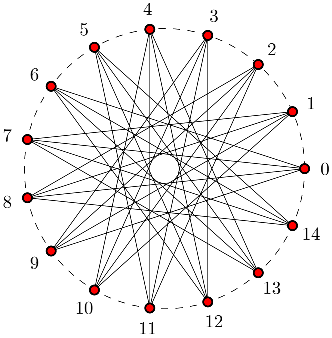

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A homomorphism from G to $K_{s/r}$ is a circular (s, r)-colouring of G.

Circular chromatic number

$$\chi_c(G) = \inf\{\frac{s}{r} : s, r \in \mathbb{N}, \hom(G, K_{s/r}) > 0\}.$$

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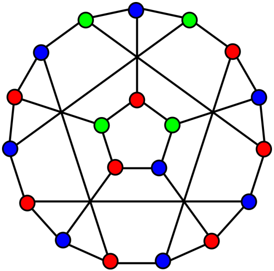

Proper 3-colouring of flower snark J_5

Figure by Rocchini, Wikimedia Commons

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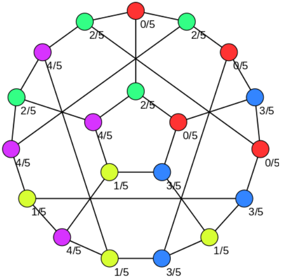

Circular $(5, 2)$-colouring of flower snark J_5

Figure by Koko90, Wikimedia Commons

Example

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Let $\mathbf{A}_k = \mathbf{T}_s \oplus \mathbf{T}_k$. Vertices of $K_{sk/rk}$: elements (a, b) of A_k^2 such that $a \in \mathbf{T}_s$ and $b \in \mathbf{T}_k$. Vertex (a, b) adjacent to vertex (a', b') if

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Proposition

For graph G and integers $s \geq 2r$, the number of circular (sk, rk)-colourings of G is polynomial in k.

Conjecture

All strongly polynomial sequences of graphs (H_k) such that $H_k \subseteq_{\mathrm{ind}} H_{k+1}$ can be obtained by QF interpretation of a "basic sequence" (disjoint union of transitive tournaments of size polynomial in k with unary relations).

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Theorem (G., Nešetřil, Ossona de Mendez , 2014+)

A sequence (H_k) of graphs of uniformly bounded degree is a strongly polynomial sequence if and only if it is a QF-interpretation of a basic sequence.

- When is $\hom(G, \mathrm{Cayley}(A_k, B_k))$ a fixed polynomial (dependent on G) in $|A_k|, |B_k|$, where $B_k = -B_k \subseteq A_k$?

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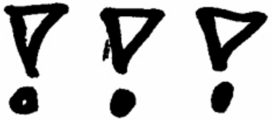
- Which strongly polynomial sequences of graphs give matroid invariants (suitably scaled like the chromatic polynomial)?



Beyond polynomials? Rational generating functions

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Beyond polynomials? Algebraic generating functions

- For sequence of odd graphs $O_k = J_{2k-1,k-1,\{0\}}$

$$\sum_k \hom(G, O_k) t^k$$

is algebraic (e.g. $\frac{1}{2}(1-4t)^{-\frac{1}{2}}$ when $G = K_1$).

Three papers

- P. de la Harpe and F. Jaeger, Chromatic invariants for finite graphs: theme and polynomial variations, *Lin. Algebra Appl.* **226–228** (1995), 687–722

  Defining graphs invariants from counting graph homomorphisms. Examples. Basic constructions.

- D. Garijo, A. Goodall, J. Nešetřil, Polynomial graph invariants from homomorphism numbers. 40pp. arXiv: 1308.3999 [math.CO]
  Further examples. New construction using tree representations of graphs.

- A. Goodall, J. Nešetřil, P. Ossona de Mendez, Strongly polynomial sequences as interpretation of trivial structures. 21pp. arXiv:1405.2449 [math.CO].

  General relational structures: counting satisfying assignments for quantifier-free formulas. Building new polynomial invariants by interpretation of "trivial" sequences of marked tournaments.

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Theorem (G., Nešetřil, Ossona de Mendez , 2014+)

If (H_k) *is strongly polynomial then there are only finitely many terms belonging to a quasi-random sequence of graphs.*