Constructing the Tutte polynomial via polyhedral or algebraic geometry

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11 July 2015

Overview

The principal work reported here is joint with David Speyer.

- Matroids and polytopes
- Dramatis personae of varieties
- The algebro-geometric approach to Tutte
- Proof technique: the polyhedral approach to Tutte

The yet-unrealised hope: prove some inequalities?

Matroids

Definition

A matroid M on finite ground set E is a system $\mathcal{B}(M) \subseteq 2^E$ of bases satisfying

- $\mathcal{B}(M) \neq \emptyset$;
- If $A, B \subseteq \mathcal{B}(M)$ and $b \in B \setminus A$, there exists $a \in A \setminus B$ such that $A \setminus \{a\} \cup \{b\}$ is a basis.

The rank $r(M)$ is the size of each basis.

Example (Graphic matroids)

A graph G yields a matroid on $E(G)$ whose bases are the spanning forests of G (i.e. a spanning tree in each component).

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Matroids II

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Example (Linearly realisable matroids)

A vector configuration E yields a matroid whose bases are the bases of the space they span.

Graphical matroids are linear over any field:
edge $(u, v) \rightsquigarrow$ vector $e_u - e_v$.

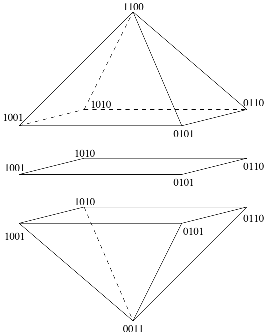
Matroid polytopes

Def./Thm. (Edmonds; Gel'fand-Goresky-MacPherson-Serganova)

A matroid M on the set $[n]$ is a polytope $P(M) \subseteq \mathbb{R}^n$ such that

- *every vertex of $P(M)$ lies in $\{0,1\}^n$ ($\rightsquigarrow$ bases);*
- *every edge of $P(M)$ is parallel to $e_j - e_i$ for some $i, j \in [n]$.*

The edges are the exchanges between the bases.



$P(M)$ lies in $\left\{ \sum_{i=1}^{n} x_i = r(M) \right\}$.

The hypersimplex is the biggest matroid polytope,

$$[0,1]^n \cap \left\{ \sum x_i = r \right\} =: P(U_{r,n}).$$

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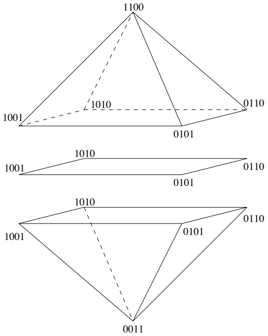

Varieties

We work over $\mathbb{C}$.

Definition

The Grassmannian is

$$G(r,n) = \{\text{configs of } n \text{ vectors spanning } \mathbb{C}^r\}/GL_r$$
$$= \{r\text{-dimensional subspaces of } \mathbb{C}^n\}.$$

An algebraic torus $T := (\mathbb{C}^*)^n$ acts on $G(r,n)$ by scaling the vectors.

If $x_M \in G(r,n)$ is a linear realisation of M,
the orbit closure $\overline{Tx_M} \subseteq G(r,n)$ is the toric variety of $P(M)$:

that is, it's the closure of $\{(t^{v_1} : \cdots : t^{v_n})\}$ where the v_i are the vertices of $P(M)$.

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K-theory

Let X be smooth.
Algebraic K-theory assigns a ring $K^0(X)$ to each variety X.

For each subvariety Y or vector bundle $\mathcal{E}$ on X
there is an element $[Y]$ or $[\mathcal{E}]$ of $K^0(X)$.

Definition

The K-theory $K^0(X)$ of a variety X is the Grothendieck group

$$\frac{\mathbb{Z}\{\text{vector bundles on } X\}}{(B = A + C \quad \text{if} \quad 0 \to A \to B \to C \to 0)}.$$

Tensor product makes it a ring.

K-theory classes can be pushed and pulled along morphisms.

Proofs of our result will use a refined invariant, the T-equivariant K-theory $K^0_T(X)$, in which only equivariant objects have classes.

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Constructing Tutte

The flag variety, of chains of subspaces in $\mathbb{C}^n$ of given dimensions, has maps extracting spaces from the chain:

$$G(r,n) \xleftarrow{\ \pi\ } \mathrm{Flag}(1, r, n-1; n) \xrightarrow{\ \rho\ } \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$$

$G(r,n)$ bears the line bundle $\mathcal{O}(1)$ whose fiber at the point representing $V \subseteq \mathbb{C}^n$ is $\bigwedge^r V$.

Theorem (F-Speyer)

If M is represented by the point $x_M \in G(r,n)$ then the Tutte polynomial of M is

$$\rho_* \pi^* ([\overline{x_M T}] \cdot [\mathcal{O}(1)]) \in K^0(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}) \cong \mathbb{Z}[x,y]/(x^n, y^n).$$

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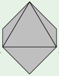
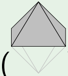
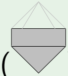
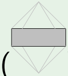
Nonrealisable matroids?

Nonrealisable matroids also have a class corresponding to $[\overline{x_M T}]$, even though they lack analogues of x_M or $\overline{x_M T}$.

Given a linear relation among indicator functions $\mathbb{1}(P(M))$ of realisable matroid polytopes, $[x_M T]$ satisfies the same relation.

Example

$\mathbb{1}($  $) - \mathbb{1}($  $) - \mathbb{1}($  $) + \mathbb{1}($  $) = 0.$

Corollary

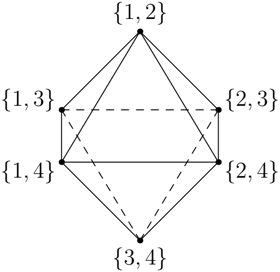
Linear relations between indicator functions give linear relations between Tutte polynomials.

Technique: Equivariant localization (Atiyah-Bott, Goresky-Kottwitz-MacPherson, ...)

If X is nice and a big enough torus T acts on it, e.g. $X = G(r, n)$, then $K^0_T(X)$ is the ring of certain vertex labellings of its moment graph $\Gamma(X)$ with Laurent polynomials in $t_1, \ldots, t_n$.

$\Gamma(G(r, n))$ is the 1-skeleton of the hypersimplex.

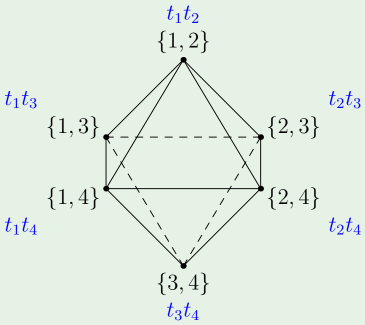
The allowable labellings are those where the labels at vertices v_S and $v_{S\cup\{a\}\setminus\{b\}}$ differ by a multiple of $t_b - t_a$.



Some K-classes

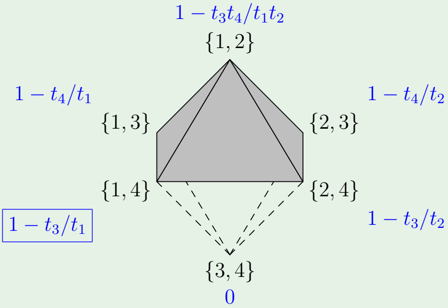
Example

$$[\mathcal{O}(1)]^T(v_S) = t^S := \prod_{i \in S} t_i.$$



Example

This is $[\overline{x_M T}]$ for the matroid M depicted.



Computing classes for matroids: enter polytopes

The Laurent polynomials for $[\overline{x_M T}]$ are numerators of lattice point generating functions of cones on $P(M)$.

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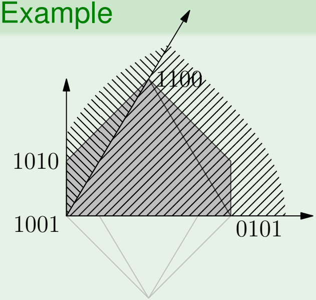

$\mathrm{Cone}_{\{1,4\}}(M)$ is simplicial, with g.f.

$$\frac{1}{(1-t_2/t_1)(1-t_2/t_4)(1-t_3/t_4)}.$$

The "expected" denominator is $\prod_{i\in S, j\notin S}(1-t_j/t_i)$.

This also works without any realisability assumption.

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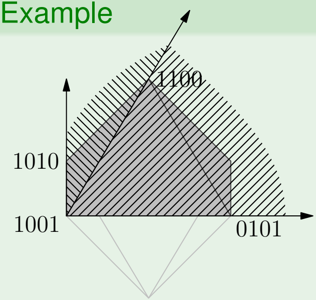

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Combinatorialising the whole picture

Let M be a matroid.

For each basis A of M, construct the cone based at $e_A = \sum_{i \in A} e_i$ and whose rays pass through e_B for every basis $B = A \setminus \{a\} \cup \{b\}$.

Let $f_A(t_1, \ldots, t_n)$ be its lattice point enumerator.

Then

$$[x^p y^q] T_M(x, y) = \left[\sum_{a,b=1}^{n} \sum_{\substack{A \in \mathcal{B}(M) \\ a \in A, b \notin A}} \frac{f_A \cdot \frac{t_{p+1}}{t_a} \prod_{i=p+2}^{n} (1 - \frac{t_i}{t_a}) \cdot \frac{t_b}{t_{q+1}} \prod_{j=q+2}^{n} (1 - \frac{t_b}{t_j})}{\prod_{i \in A \setminus \{a\}} (1 - \frac{t_i}{t_a}) \cdot \prod_{j \in [n] \setminus A \setminus \{b\}} (1 - \frac{t_b}{t_j})} \right]_{t=1}.$$

Closing remarks

A simpler way

Amanda Cameron and I have found a more direct way to extract the Tutte polynomial from polyhedral geometry.

This yields a change of variables under which the coefficients count something; we hope this will give new interesting inequalities for Tutte.

Check it out at the poster session!

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