Krushkal polynomial of graphs on surfaces

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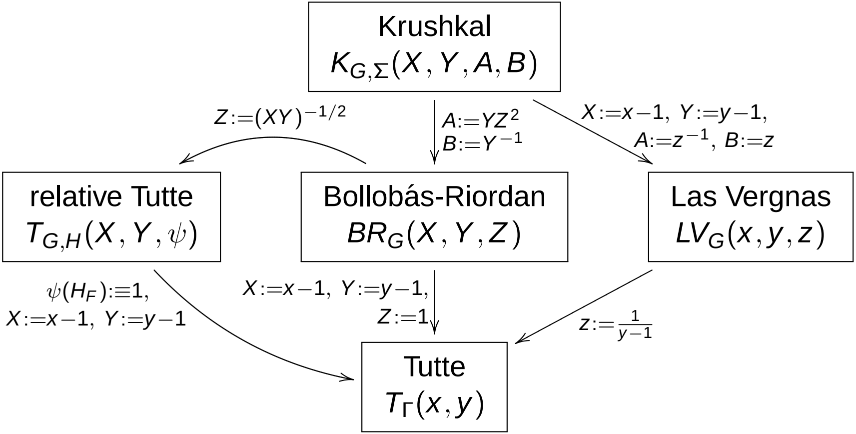
Workshop on the Tutte Polynomial
Royal Holloway University

Sunday, July 12, 2015
10:00–10:30am

Polynomials of graphs on surfaces.



Krushkal polynomial.

Definition. Let G be a graph embedded into a surface Σ.

$$K_{G,\Sigma}(X,Y,A,B) := \sum_{F\subseteq G} X^{k(F)-k(G)} Y^{k(\Sigma\setminus F)-k(\Sigma)} A^{g(F)} B^{g^{\perp}(F)},$$

where the sum runs over all spanning subgraphs considered as ribbon graphs;
$k(F)$ stands for the number of connected components of the surface F;
the parameters $g(F)$ and $g^{\perp}(F)$ stand for the genera of surfaces F and $\Sigma\setminus F$.
For non-orientable surfaces they are equal to one half of the number of Möbius bands glued into spheres to represent the surfaces.

Topological meaning of exponents.

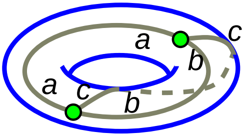
$$\begin{aligned} k(\Sigma\setminus F) - k(\Sigma) &= \dim(\ker(H_1(F;\mathbb{Z}_2) \to H_1(\Sigma;\mathbb{Z}_2))), \\ s(F) &= \dim H_1(\widetilde{F};\mathbb{Z}_2), \\ s^{\perp}(F) &= \dim H_1(\widetilde{\Sigma\setminus F};\mathbb{Z}_2), \end{aligned}$$

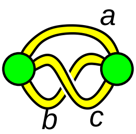
where $\widetilde{F}$ and $\widetilde{\Sigma\setminus F}$ are the surfaces obtained by gluing a disc to each boundary component of surfaces F and $\Sigma\setminus F$.

Properties.

$$K_{G,\Sigma} = \begin{cases} K_{G/e,\Sigma} + K_{G-e,\Sigma} & \text{if } e \text{ is ordinary, that is neither a bridge nor a loop,} \\ (1+X)\cdot K_{G/e,\Sigma} & \text{if } e \text{ is a bridge.} \\ (1+Y)\cdot BR_{G-e,\Sigma} & \text{if } e \text{ is a separable loop, the one whose removal together with its vertex separates the surface } \Sigma. \end{cases}$$

$K_{G_1\sqcup G_2,\Sigma_1\sqcup S_2} = K_{G_1,\Sigma_1}\cdot K_{G_2,\Sigma_2}$, where $\sqcup$ is a disjoint union.

$G =$ 

$=$ 

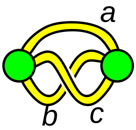
$\varkappa = k(\Sigma \setminus F) - k(\Sigma)$

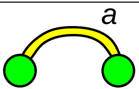
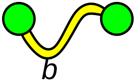
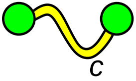
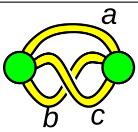
F	$\emptyset$	$\{a\}$	$\{b\}$	$\{a,b\}$	$\{c\}$	$\{a,c\}$	$\{b,c\}$	$\{a,b,c\}$
$k(F)$	2	1	1	1	1	1	1	1
$\varkappa(F)$	0	0	0	0	0	0	0	0
$g(F)$	0	0	0	0	0	0	0	1
$g^{\perp}(F)$	1	1	1	0	1	0	0	0
$K_{G,\Sigma}$	XB	B	B	1	B	1	1	A

$$K_{G,\Sigma} = 3 + 3B + XB + A.$$

Quasi-trees.

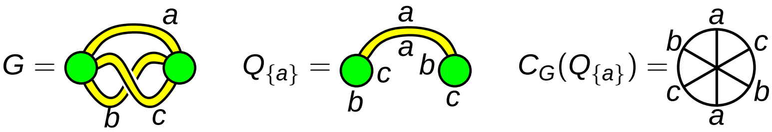
Definition. A *quasi-tree* is a ribbon graph with one boundary component.

$G =$ 

$Q_{\{a\}}$	$Q_{\{b\}}$	$Q_{\{c\}}$	$Q_{\{a,b,c\}}$
			

Quasi-tree activities. Chord diagrams.

A round trip along the boundary component of Q passes the boundary arcs of each edge-ribbon twice. A *chord diagram* $C_G(Q)$ consists of a circle corresponding to the boundary of Q and chords connecting the pairs of arcs corresponding to the same edge-ribbon.



Quasi-tree activities.

Let $\prec$ be a total order of edges $E(G)$.

Definition [A.Champanerkar, I.Kofman, N.Stoltzfus].

An edge is called *live* if the corresponding chord is smaller than any chord intersecting it relative to the order $\prec$. Otherwise it is called *dead*.

For plane graphs G a spanning quasi-tree is a tree and the notion of *live/dead* coincides with the classical Tutte's notion of *active/inactive*.

In the example above the edge a is live and the edges b and c are dead relative to the order $a \prec b \prec c$ for all four quasi-trees.

Quasi-tree expansion of the Krushkal polynomial.

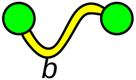
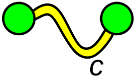
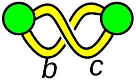
Theorem [C.Butler].
For a ribbon graph G, the Krushkal polynomial has the following expansion over the set of quasi-trees.

$$K_G(X,Y,A,B) = \sum_{Q\in\mathcal{Q}_G} A^{g(F(Q))} T_Q \cdot B^{g(F(Q^*))} T_{Q^*} \ ,$$

where $T_Q = T_{\Gamma(Q)}(X+1, A+1)$ *and* $T_{Q^*} = T_{\Gamma(Q^*)}(Y+1, B+1)$ *stand for the classical Tutte polynomial of abstract graphs* $\Gamma(Q)$ *and* $\Gamma(Q^*)$.

Definition.

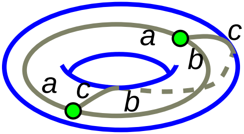
- $F(Q)$ is a spanning ribbon subgraph of Q obtained by deleting the internally live (orientable) edges of Q;
- $\Gamma(Q)$ is a usual abstract (not embedded) graph whose vertices are the connected components of $F(Q)$ and edges are the internally live (orientable) edges of Q.

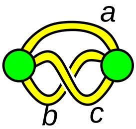
Q	$Q_{\{a\}}$	$Q_{\{b\}}$	$Q_{\{c\}}$	$Q_{\{a,b,c\}}$
$F(Q)$				
$\Gamma(Q)$				

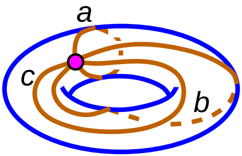
Dual graphs.

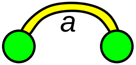
Let $G^\star$ be the usual Poincaré dual graph ribbon graph to G, regarded as a graph cellularly embedded into the surface $\Sigma = \widetilde{G}$.

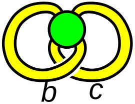
A spanning subgraph $F \subseteq G$ determines a spanning subgraph $F^* \subseteq G^\star$ containing all edges of $G^\star$ which do not intersect edges of F.

$G =$ 

$=$ 

$G^\star =$ 

$Q_{\{a\}} =$ 

$Q^*_{\{a\}} =$ 

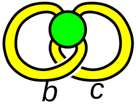
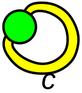
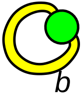
Dual quasi-trees.

- The spanning subgraphs F and F^* have common boundary and their gluing along this common boundary gives the whole surface Σ.
- If Q is a spanning quasi-tree for G, then subgraph Q^* is a quasi-tree for $G^\star$.
- These quasi-trees have the same chord diagrams, $C_G(Q) = C_{G^\star}(Q^*)$.
- The natural bijection of edges of G and $G^\star$ leads to the total order $\prec^\star$ on edges of $G^\star$ induced by $\prec$.
- The property of an edge of being live/dead relative to Q is preserved by the bijection to the same property relative to Q^*.
- The property of being internal/external is changed to the opposite.

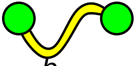
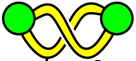
$F(Q^*)$ and $\Gamma(Q^*)$.

Definition.

- $F(Q^*)$ is a spanning ribbon subgraph of Q^* obtained by deleting the internally live (orientable) edges of Q^*;
- $\Gamma(Q^*)$ is an abstract graph whose vertices are the connected components of $F(Q^*)$ and edges are the internally live (orientable) edges of Q^*.

Q^*	$Q^*_{\{a\}}$	$Q^*_{\{b\}}$	$Q^*_{\{c\}}$	$Q^*_{\{a,b,c\}}$
$F(Q^*)$				
$\Gamma(Q^*)$				

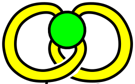
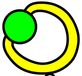
Quasi-tree expansion.

Q	$Q_{\{a\}}$	$Q_{\{b\}}$	$Q_{\{c\}}$	$Q_{\{a,b,c\}}$
$F(Q)$				
$\Gamma(Q)$				
$A^{g(F(Q))}$	1	1	1	1
T_Q	$X+1$	1	1	$A+1$

$$K_G(X,Y,A,B) = \sum_{Q\in\mathcal{Q}_G} A^{g(F(Q))}T_Q \cdot B^{g(F(Q^*))}T_{Q^*} \ ,$$

where $T_Q = T_{\Gamma(Q)}(X+1, A+1)$ and $T_{Q^*} = T_{\Gamma(Q^*)}(Y+1, B+1)$

Quasi-tree expansion. Dual part.

Q^*	$Q^*_{\{a\}}$	$Q^*_{\{b\}}$	$Q^*_{\{c\}}$	$Q^*_{\{a,b,c\}}$
$F(Q^*)$				
$\Gamma(Q^*)$				
$B^{g(F(Q^*))}$	B	1	1	1
T_{Q^*}	1	$B+1$	$B+1$	1

$$K_G = (X+1)B + (B+1) + (B+1) + (A+1) = XB + A + 3B + 3,$$

References.

- C. Butler, *A quasi-tree expansion of the Krushkal polynomial*, Preprint `arXiv:1205.0298[math.CO]`.
- A. Champanerkar, I. Kofman, N. Stoltzfus, *Quasi-tree expansion for the Bollobás-Riordan-Tutte polynomial*, Bull.Lond.Math.Soc., **43**(5) (2011) 972–984.

THANK YOU!