Cyclic Flats of Matroids and their connections to Tutte Polynomials and Other Matroid Invariants

Joseph E. Bonin

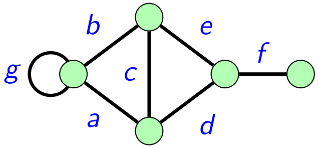
The George Washington University

These slides are available at
http://home.gwu.edu/∼jbonin/

A matroid M consists of a finite set $E(M)$ and function $r : 2^{E(M)} \to \mathbb{Z}$ (the rank function) such that:

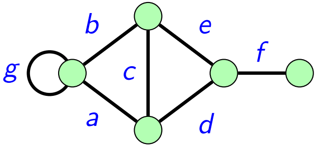
- $0 \le r(X) \le |X|$ for all $X \subseteq E(M)$,
- if $X \subseteq Y \subseteq E(M)$, then $r(X) \le r(Y)$, and
- $r(X) + r(Y) \ge r(X \cap Y) + r(X \cup Y)$ for all $X, Y \subseteq E(M)$. (submodularity)

E.g., for a graph (V, E) and set $X \subseteq E$, let $r(X)$ be the number of edges in a maximal forest in (V, X).



E.g., $r(c, e) = r(c, d, e, g) = 2, \quad r(a, b, e) = r(a, b, c, d, e, g) = 3$

Matroid basics: some terms



A set X is independent if $r(X) = |X|$. E.g., $\{b, f\}$ and $\{a, b, e\}$.

A circuit is a minimal dependent set. E.g., $\{g\}$, $\{a, b, c\}$, and $\{a, b, d, e\}$.

Loops are elements in singleton circuits. E.g., g.

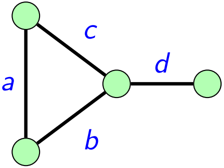
Coloops are elements that are in no circuits. E.g., f.

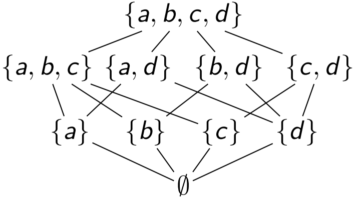
A set X is a flat of M if $r(X \cup y) > r(X)$ for all $y \in E(M) - X$.
E.g., $\{g\}$, $\{d, g\}$, $\{c, d, e, g\}$, and E.

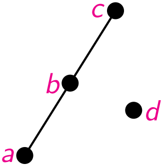
Matroid basics: flats

A set X is a **flat** if $r(X \cup y) > r(X)$ for all $y \in E(M) - X$.

Under inclusion, the flats form a geometric lattice.
(atomic and semimodular)





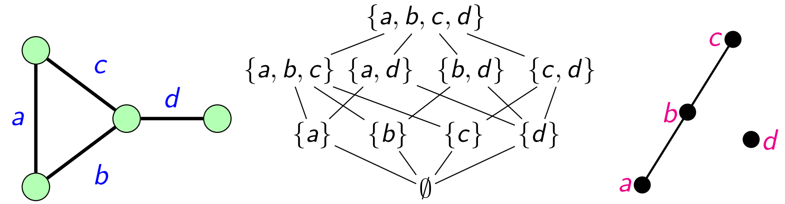


The set of flats determines M.

Matroid basics: flats

A set X is a **flat** if $r(X \cup y) > r(X)$ for all $y \in E(M) - X$.

Under inclusion, the flats form a geometric lattice.
(atomic and semimodular)



The set of flats determines M.

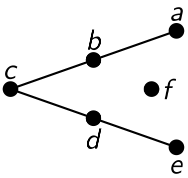
Equivalent formulations of the closure, $\mathrm{cl}(X)$, of $X \subseteq E(M)$:

- the least flat that contains X,
- $\{y : r(X) = r(X \cup y)\}$,
- $X \cup \{y :$ there is a circuit C with $y \in C \subseteq X \cup y\}$.

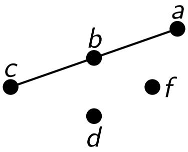
E.g., $\mathrm{cl}(a, b) = \{a, b, c\}$.

Matroid basics: restriction

The restriction, $M|X$, consists of X and the restriction of r to its subsets.



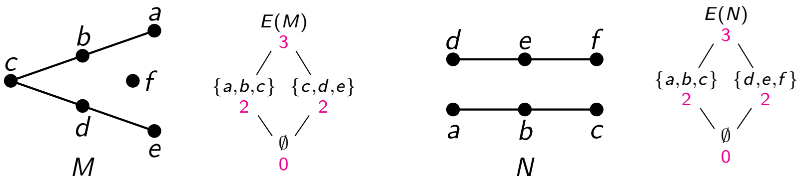
M



$M|\{a, b, c, d, f\}$

Cyclic flats

A set X in a matroid M is cyclic if X is a union of circuits of M, i.e., $M|X$ has no coloops.



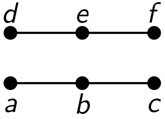
The set $\mathcal{Z}(M)$ of cyclic flats of M, ordered by inclusion, is a lattice.

Joins: $X \vee Y = \text{cl}(X \cup Y)$.

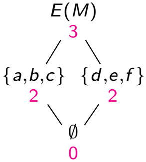
Meets: from $X \cap Y$, delete the coloops of $M|(X \cap Y)$.

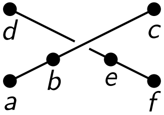
The cyclic flats alone do not determine a matroid

Many matroids may have the same cyclic flats, with different ranks.

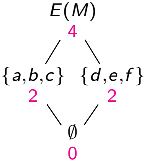


rank 3

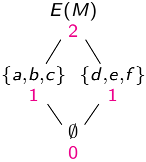




rank 4

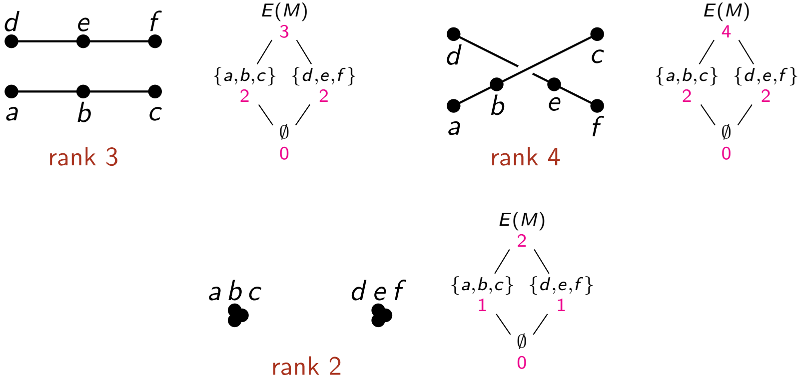


rank 2



The cyclic flats alone do not determine a matroid

Many matroids may have the same cyclic flats, with different ranks.



A matroid M is determined by $E(M)$ and the pairs $(X, r(X))$ with $X \in \mathcal{Z}(M)$.

(Brylawski, 1975)

Theorem

For $\mathcal{Z} \subseteq 2^E$ and $r : \mathcal{Z} \to \mathbb{Z}$, there is a matroid M on E with $\mathcal{Z} = \mathcal{Z}(M)$ and r is r_M restricted to $\mathcal{Z}$ iff

(Z0) $(\mathcal{Z}, \subseteq)$ *is a lattice,*

(Z1) $r(0_{\mathcal{Z}}) = 0$, *where $0_{\mathcal{Z}}$ is the least element of $\mathcal{Z}$,*

(Z2) $0 < r(Y) - r(X) < |Y - X|$ *for all $X, Y \in \mathcal{Z}$ with $X \subsetneq Y$,*

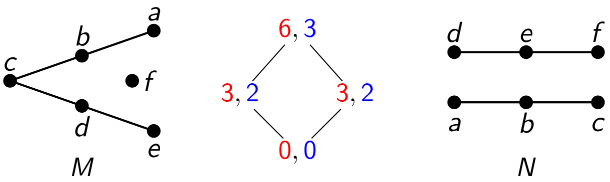
and

(Z3) $r(X) + r(Y) \geq r(X \vee Y) + r(X \wedge Y) + |(X \cap Y) - (X \wedge Y)|$ *for all $X, Y \in \mathcal{Z}$.*

(Sims, 1980; Bonin and de Mier, 2008)

The configuration of M is a 4-tuple $(L, s, \rho, |E(M)|)$ where

- L is an abstract lattice with $L \simeq \mathcal{Z}(M)$;
  say $x \mapsto X$ is an isomorphism of $L \to \mathcal{Z}(M)$,
- $s : L \to \mathbb{Z}$ with $s(x) = |X|$,
- $\rho : L \to \mathbb{Z}$ with $\rho(x) = r(X)$. (Eberhardt, 2014)



The cyclic flats themselves are not recorded,
so non-isomorphic matroids may have the same configuration.

Theorem

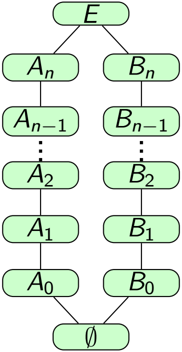
The Tutte polynomial of M is determined by its configuration.

(Eberhardt, 2014)

Many non-isomorphic matroids can have the same configuration.

O. Giménez' example:

Fix $\pi \in S_n$, the group of permutations of $\{1, 2, \ldots, n\}$.



Fix four disjoint sets

A_0 with $|A_0| = n + 2$ and $r(A_0) = n + 1$,

B_0 with $|B_0| = n + 3$ and $r(B_0) = n + 1$,

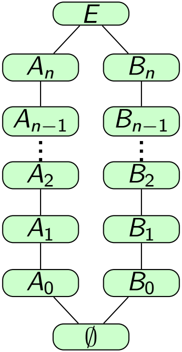
$\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$.

Let E be their union, so $|E| = 4n + 5$.

Many non-isomorphic matroids can have the same configuration.

O. Giménez' example:

Fix $\pi \in S_n$, the group of permutations of $\{1, 2, \ldots, n\}$.



Fix four disjoint sets

A_0 with $|A_0| = n + 2$ and $r(A_0) = n + 1$,

B_0 with $|B_0| = n + 3$ and $r(B_0) = n + 1$,

$\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$.

Let E be their union, so $|E| = 4n + 5$.

For $i \in [n]$, set

$A_i = A_{i-1} \cup \{x_i, y_i\}$ with $r(A_i) = n + i + 1$,

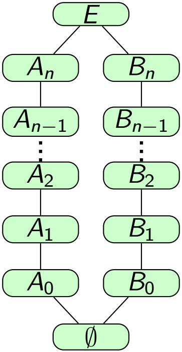
$B_i = B_{i-1} \cup \{x_i, y_{\pi(i)}\}$ with $r(B_i) = n + i + 1$,

and set $r(E) = 2n + 2$.

Many non-isomorphic matroids can have the same configuration.

O. Giménez' example:

Fix $\pi \in S_n$, the group of permutations of $\{1, 2, \ldots, n\}$.



Fix four disjoint sets

A_0 with $|A_0| = n + 2$ and $r(A_0) = n + 1$,
B_0 with $|B_0| = n + 3$ and $r(B_0) = n + 1$,
$\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$.

Let E be their union, so $|E| = 4n + 5$.

For $i \in [n]$, set
$A_i = A_{i-1} \cup \{x_i, y_i\}$ with $r(A_i) = n + i + 1$,
$B_i = B_{i-1} \cup \{x_i, y_{\pi(i)}\}$ with $r(B_i) = n + i + 1$,
and set $r(E) = 2n + 2$.

These $n!$ matroids of rank $2n + 2$ are non-isomorphic, have the same configuration, and so have the same Tutte polynomial.

Sketch of a new proof of Eberhardt's result: reductions

$$T(M; x, y) = \sum_{A \subseteq E(M)} (x-1)^{r(M)-r(A)} (y-1)^{|A|-r(A)}$$
$$= \sum_{i \leq j} |E_{i,j}| \, (x-1)^{r(M)-i} (y-1)^{j-i}.$$

where $E_{i,j} = \{X \subseteq E(M) \,:\, r(X) = i, \, |X| = j\}$.

Goal: show that $|E_{i,j}|$ is determined by the configuration.

Sketch of a new proof of Eberhardt's result: reductions

$$T(M;x,y) = \sum_{A\subseteq E(M)} (x-1)^{r(M)-r(A)}(y-1)^{|A|-r(A)}$$
$$= \sum_{i\leq j} |E_{i,j}|\,(x-1)^{r(M)-i}(y-1)^{j-i}.$$

where $E_{i,j} = \{X \subseteq E(M) : r(X) = i,\, |X| = j\}$.

Goal: show that $|E_{i,j}|$ is determined by the configuration.

It suffices to treat matroids without loops and coloops.

Induct on $|\mathcal{Z}(M)|$. The case $|\mathcal{Z}(M)| = 2$ is easy (M is uniform: $r(X) = \min(|X|, r)$), so assume $|\mathcal{Z}(M)| > 2$.

Sketch of a new proof of Eberhardt's result: reductions

$$T(M;x,y) = \sum_{A\subseteq E(M)} (x-1)^{r(M)-r(A)}(y-1)^{|A|-r(A)}$$
$$= \sum_{i\leq j} \left|E_{i,j}\right| (x-1)^{r(M)-i}(y-1)^{j-i}.$$

where $E_{i,j} = \{X \subseteq E(M) \,:\, r(X)=i,\, |X|=j\}$.

Goal: show that $|E_{i,j}|$ is determined by the configuration.

It suffices to treat matroids without loops and coloops.

Induct on $|\mathcal{Z}(M)|$. The case $|\mathcal{Z}(M)| = 2$ is easy (M is uniform: $r(X) = \min(|X|, r)$), so assume $|\mathcal{Z}(M)| > 2$.

Reduction: treat only $i < \min(r,j)$ since

$$|E_{r,j}| = \binom{n}{j} - \sum_{i<r} |E_{i,j}| \qquad \text{and} \qquad |E_{j,j}| = \binom{n}{j} - \sum_{i<j} |E_{i,j}|.$$

Let $\mathcal{Z}'(M) = \mathcal{Z}(M) - \{\mathrm{cl}(\emptyset), E(M)\}$.

For $F \in \mathcal{Z}'(M)$ and $i < \min(r, j)$, set

$D_F = \{X : X \in E_{i,j} \text{ and } \mathrm{cl}(X \cap F) = F\}$.

Let $\mathcal{Z}'(M) = \mathcal{Z}(M) - \{\text{cl}(\emptyset), E(M)\}$.

For $F \in \mathcal{Z}'(M)$ and $i < \min(r, j)$, set
$D_F = \{X : X \in E_{i,j} \text{ and } \text{cl}(X \cap F) = F\}$.

Since $i < \min(r, j)$, any $X \in E_{i,j}$ contains a circuit C with $\text{cl}(C) \in \mathcal{Z}'(M)$, so

$$E_{i,j} = \bigcup_{F \in \mathcal{Z}'(M)} D_F.$$

Sketch of a new proof of Eberhardt's result: PIE

Let $\mathcal{Z}'(M) = \mathcal{Z}(M) - \{\mathrm{cl}(\emptyset), E(M)\}$.

For $F \in \mathcal{Z}'(M)$ and $i < \min(r, j)$, set
$D_F = \{X : X \in E_{i,j} \text{ and } \mathrm{cl}(X \cap F) = F\}$.

Since $i < \min(r, j)$, any $X \in E_{i,j}$ contains a circuit C with $\mathrm{cl}(C) \in \mathcal{Z}'(M)$, so

$$E_{i,j} = \bigcup_{F \in \mathcal{Z}'(M)} D_F.$$

So, by inclusion/exclusion,

$$|E_{i,j}| = \sum_{S \subseteq \mathcal{Z}'(M),\, S \neq \emptyset} (-1)^{|S|+1} \Big| \bigcap_{F \in S} D_F \Big|.$$

Sketch of a new proof of Eberhardt's result: simplification to chains

Key: $F_1, F_2 \in \mathcal{Z}'(M)$ are incomparable, then $D_{F_1} \cap D_{F_2} \subseteq D_{F_1 \vee F_2}$.

If $X \in D_{F_1} \cap D_{F_2}$, then $\mathrm{cl}(X \cap F_1) = F_1$ and $\mathrm{cl}(X \cap F_2) = F_2$, so $\mathrm{cl}(X \cap \mathrm{cl}(F_1 \cup F_2)) = \mathrm{cl}(F_1 \cup F_2)$.

Thus, if $F_1, F_2 \in S \subseteq \mathcal{Z}'(M)$ and $S' = S \triangle \{F_1 \vee F_2\}$, then

$$\bigcap_{F \in S} D_F = \bigcap_{F \in S'} D_F.$$

Sketch of a new proof of Eberhardt's result: simplification to chains

Key: $F_1, F_2 \in \mathcal{Z}'(M)$ are incomparable, then $D_{F_1} \cap D_{F_2} \subseteq D_{F_1 \vee F_2}$.

If $X \in D_{F_1} \cap D_{F_2}$, then $\mathrm{cl}(X \cap F_1) = F_1$ and $\mathrm{cl}(X \cap F_2) = F_2$, so $\mathrm{cl}(X \cap \mathrm{cl}(F_1 \cup F_2)) = \mathrm{cl}(F_1 \cup F_2)$.

Thus, if $F_1, F_2 \in S \subseteq \mathcal{Z}'(M)$ and $S' = S \triangle \{F_1 \vee F_2\}$, then

$$\bigcap_{F \in S} D_F = \bigcap_{F \in S'} D_F.$$

Since $(-1)^{|S|} = -(-1)^{|S'|}$, the corresponding terms could cancel in the sum from PIE.

We can match such potential cancellations, leaving

$$|E_{i,j}| = \sum_{\substack{\text{nonempty chains} \\ S \subseteq \mathcal{Z}'(M)}} (-1)^{|S|+1} \left| \bigcap_{F \in S} D_F \right|.$$

Sketch of a new proof of Eberhardt's result: treat chains

$$\left|E_{i,j}\right| = \sum_{\substack{\text{nonempty chains} \\ S \subseteq \mathcal{Z}'(M)}} (-1)^{|S|+1} \left| \bigcap_{F \in S} D_F \right|.$$

For a chain $S = \{F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_p\}$, we can compute $\left|\bigcap_{F \in S} D_F\right|$ using the minors

$$M|F_1, \qquad M|F_k/F_{k-1}, \quad \text{and} \quad M/F_p$$

that correspond to the intervals

$$[\emptyset, F_1], \qquad [F_{k-1}, F_k], \quad \text{and} \quad [F_p, E(M)]$$

in $\mathcal{Z}(M)$, where $2 \leq k \leq p$.

Sketch of a new proof of Eberhardt's result: treat chains

$$|E_{i,j}| = \sum_{\substack{\text{nonempty chains} \\ S \subseteq \mathcal{Z}'(M)}} (-1)^{|S|+1} \Big| \bigcap_{F \in S} D_F \Big|.$$

For a chain $S = \{F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_p\}$, we can compute $\left|\bigcap_{F \in S} D_F\right|$ using the minors

$$M|F_1, \qquad M|F_k/F_{k-1}, \quad \text{and} \quad M/F_p$$

that correspond to the intervals

$$[\emptyset, F_1], \qquad [F_{k-1}, F_k], \quad \text{and} \quad [F_p, E(M)]$$

in $\mathcal{Z}(M)$, where $2 \leq k \leq p$.

Each of these minors has fewer cyclic flats than M.
We get their configurations from that of M.
That allows us to find $|E_{i,j}|$ from the configuration of M. $\square$

Let M be a rank-r matroid on $\{1, 2, \ldots, n\}$.

For $\pi \in S_n$, the rank sequence $\underline{r}(\pi) = (r_1, r_2, \ldots, r_n)$ is given by $r_1 = r(\{\pi(1)\})$ and for $j \geq 2$,

$$r_j = r(\{\pi(1), \pi(2), \ldots, \pi(j)\}) - r(\{\pi(1), \pi(2), \ldots, \pi(j-1)\}).$$

So $r_j \in \{0, 1\}$ and $\{\pi(j) \ : \ r_j = 1\}$ is a basis of M.

Let M be a rank-r matroid on $\{1, 2, \ldots, n\}$.

For $\pi \in S_n$, the rank sequence $\underline{r}(\pi) = (r_1, r_2, \ldots, r_n)$ is given by $r_1 = r(\{\pi(1)\})$ and for $j \geq 2$,

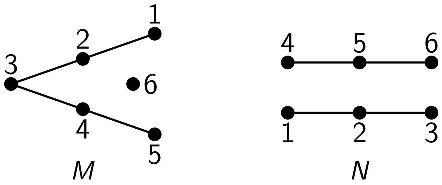
$$r_j = r(\{\pi(1), \pi(2), \ldots, \pi(j)\}) - r(\{\pi(1), \pi(2), \ldots, \pi(j-1)\}).$$

So $r_j \in \{0, 1\}$ and $\{\pi(j) \,:\, r_j = 1\}$ is a basis of M.

To a sequence $\underline{r}$ of r 1's and $n - r$ 0's, associate a variable $[\underline{r}]$.

The $\mathcal{G}$-invariant is $\mathcal{G}(M) = \sum_{\pi \in S_n} [\underline{r}(\pi)]$. (A reformulation.)

An example



Two rank sequences:
111000 if $\{\pi(1), \pi(2), \pi(3)\}$ a basis;

there are $\left(\binom{6}{3} - 2\right) \cdot 3! \cdot 3! = 648$ such $\pi \in S_6$;

110100 otherwise;

there are $2 \cdot 3! \cdot 3! = 72$ such $\pi \in S_6$.

Thus, $\mathcal{G}(M) = \mathcal{G}(N) = 648\,[111000] + 72\,[110100]$.

Theorem

The Tutte polynomial $T(M; x, y)$ is a specialization of $\mathcal{G}(M)$.

(Derksen, 2009)

A **flag** of a rank-r matroid is a maximal chain of flats

$$\mathrm{cl}(\emptyset) = X_0 < X_1 < X_2 < \cdots < X_{r-1} < X_r = E(M),$$

where $r(X_j) = j$.

Theorem

$\mathcal{G}(M)$ is equivalent to the data

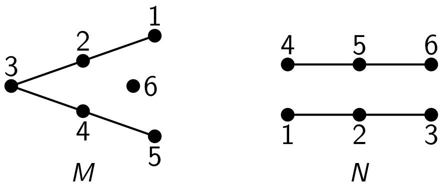
$$|X_0|,\ |X_1 - X_0|,\ |X_2 - X_1|, \ldots,\ |X_r - X_{r-1}|$$

for all flags of M.

(Kung, 2015)

Having $\mathcal{G}(M)$ is equivalent to having all the flag data

$$|X_0|,\ |X_1 - X_0|,\ |X_2 - X_1|, \ldots, |X_r - X_{r-1}|.$$



Two sequences arise from flags:

$0, 1, 1, 4$ arises from chains that contain a 2-point line;

there are $9 \cdot 2$ such chains;

$0, 1, 2, 3$ arises from chains that contain a 3-point line;

there are $2 \cdot 3$ such chains.

Theorem

From the configuration of M, one can compute the data

$$|X_0|,\ |X_1 - X_0|,\ |X_2 - X_1|, \ldots,\ |X_r - X_{r-1}|$$

for all flags of M, and so its $\mathcal{G}$-invariant. (Bonin and Kung, 2015)

The proof is an inclusion/exclusion argument, akin to our proof of Eberhardt's result.

Theorem

From the configuration of M, one can compute the data

$$|X_0|,\ |X_1 - X_0|,\ |X_2 - X_1|, \ldots,\ |X_r - X_{r-1}|$$

for all flags of M, and so its $\mathcal{G}$-invariant. (Bonin and Kung, 2015)

The proof is an inclusion/exclusion argument, akin to our proof of Eberhardt's result.

Dowling matroids of the same rank $r > 3$ over non-isomorphic groups of the same order show that matroids with different configurations can share the same flag data, and so have the same $\mathcal{G}$-invariant.

Other areas where cyclic flats have been useful

Cyclic flats (fully dependent flats) play many important roles in the theory of transversal matroids. (Ingleton, Brualdi, Mason, Brylawski, ...)

Other areas of application:

- constructing infinite sets of intertwines for most pairs of matroids (Bonin, 2010)
- progress on the sticky matroid conjecture (Bonin, 2011)
- matroid constructions (the free product, Crapo and Schmitt, 2005; matroid splicing, Bonin and Schmitt, 2011; semidirect sums of matroids, Bonin and Kung, 2015)
- describing some excluded minors for base-orderable and strongly base-orderable matroids (Bonin and Savitsky, 2015).

Thank you for listening.